A FUNCTORIAL FORMALISM FOR QUASI-COHERENT SHEAVES ON A GEOMETRIC STACK

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ABSTRACT. A geometric stack is a quasi-compact and semi-separated algebraic stack. We prove that the quasi-coherent sheaves on the small flat topology, Cartesian presheaves on the underlying category, and comodules over a Hopf algebroid associated to a presentation of a geometric stack are equivalent categories. As a consequence, we show that the category of quasi-coherent sheaves on a geometric stack is a Grothendieck category.

We associate, in a 2-functorial way, to a 1-morphism of geometric stacks $f\colon \mathbf{X} \to \mathbf{Y}$, an adjunction $f^* \dashv f_*$ for the corresponding categories of quasi-coherent sheaves that agrees with the classical one defined for schemes. This construction is described both geometrically in terms of the small flat site and algebraically in terms of the Hopf algebroid.

CONTENTS

INTRODUCTION

If one looks at the literature for the notion of quasi-coherent sheaf on an algebraic stack, one finds a somewhat confusing situation. For starters, the

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very notion of quasi-coherent sheaf does not refer to a ringed site but is often mistook with a related notion, that of Cartesian *pre*sheaf. After [AJPV], we became interested in the properties of its derived category. This is our approach to develop a sound basis for this study.

As a starting point, the definition of quasi-coherent sheaf should be given in the setting of a ringed site by an extension of the original definition given in [EGA I, (5.1.3)]. After our research took off, we discovered that *the Stacks Project* [SP] uses this kind of definition. The main technical problem is the lack of functoriality of the *lisse-étale* site. The Stacks Project solution is to work over the big flat sites. We do not follow this path because the category of sheaves of modules over a big (ringed) site does not possess a generator. This paper was developed independently of [SP], so we have tried to refer mainly to [LMB] for the basics on stacks.

We assume always that the stacks we consider are semi-separated, *i.e.* the diagonal is an affine morphism. On schemes this hypothesis is associated to a nice cohomological behavior. The stacks that are furthermore quasi-compact are called geometric stacks in [L] and this will be our standing assumption. A useful feature of geometric stacks is that they may be represented by a purely algebraic object, a Hopf algebroid. For such a stack $\mathbf{X}$ we consider a certain variant of the small flat site, that we will denote by $\mathbf{X}_{\mathsf{fppf}}$ (see 3.5). Its associated topos is not functorial, but still, it has enough functoriality properties to allow us to develop a useful formalism. Notice that it is finer than the classical *lisse-étale* site, but has the advantage that a presentation yields a covering for every object in the small flat topology. This allows for simpler proofs and the category of quasi-coherent sheaves does not change.

In the affine case there are three ways of understanding a quasi-coherent sheaf, namely,

- as a sheaf of modules such that locally admits a presentation,
- as a cartesian presheaf[1], and
- as a module over the ring of global sections.

These three aspects are, in some sense, present also in the global case. In this paper we show that they arise quite naturally when we consider a geometric stack. For the small flat site, cartesian presheaves of modules are sheaves (in fact, for any topology coarser than the fpqc) and moreover they are the same as quasi-coherent sheaves. This is proved in Theorem 3.11. A geometric stack may be described by a Hopf algebroid, as the stackification of its associated affine groupoid scheme. We prove that there is an equivalence of categories between comodules over a Hopf algebroid and quasi-coherent sheaves on its associated geometric stack.

This equivalence is developed in several steps. In section 4, we see that quasi-coherent sheaves on a geometric stack correspond to descent data for quasi-coherent sheaves on an affine cover (Theorem 4.6). Incidentally, this is

[1]This property corresponds essentially to the fact that a localization of a module in an element may be computed as a tensor product.

the definition of quasi-coherent sheaf considered in the classic literature, *e.g.* [V1]. Once we are in the affine setting, we see that a descent datum corresponds to certain supplementary structure on its module of global sections.

By an algebraic procedure, in section 5, we show that this structure can be transformed into the structure of a comodule, establishing a further equivalence of categories. All these equivalences combined provide the equivalence between quasi-coherent sheaves on the geometric stack and comodules over the Hopf algebroid (Corollary 5.9). An analogous result is due by Hovey under a different setting, as we will discuss later. An important consequence of this equivalence is that the category of quasi-coherent sheaves on a geometric stack is Grothendieck, *i.e.* abelian, with exact filtered direct limits and a set of generators.

The motivation for the results in the present paper is to have a convenient formalism for doing cohomology. So, it is crucial to have a good functoriality. We devote section 6 to this issue. Specifically, we show that a map between geometric stacks induce a pair of adjoint functors between the corresponding categories of quasi-coherent sheaves. This construction is not straightforward because, as we remarked before, there is no underlying map of toposes, so the definition of the direct image has extra complications. Moreover, we show that these constructions are 2-functorial. One needs to check in addition that 2-cells induce a natural transformation between their associated functors, a feature of stacks.

In section 7, we express the adjunction in a purely algebraic way. The benefit of this result is to put ourselves in the most convenient setting for approaching cohomological questions. It is noteworthy that for a 1-morphism of geometric stacks $f : \mathbf{X} \to \mathbf{Y}$, the functor f_* has a nicer geometric description while its adjoint f^* is easier to describe algebraically.

In the final section we compare our functoriality formalism with the classical one. Let us recall that the category quasi-coherent sheaves on a scheme does not change if we replace the Zariski by the étale topology. Moreover, the étale topology gives a reasonable topos for a geometric Deligne-Mumford stack. So in this setting there is a couple of adjoint functors that restrict to the classical ones in the scheme case and agree with the ones just defined in the general case of a geometric stack. This is shown in Propositions 8.13, 8.14 and Corollary 8.15.

In future work, we plan to apply the results obtained here to study the cohomology of geometric stacks.

Finally, let us compare our approach with previously published results.

The classical reference [LMB] defines quasi-coherent sheaf as a mixture of Cartesian sheaf plus a local condition of quasi-coherence. Besides, it overlooks the lack of functoriality of the *lisse-étale* site. Olsson provides a remedy [O] but his pull-back functor has a complicated description through simplicial sites. Another approach is given by Hovey [Ho1], [Ho2]. His definition of

quasi-coherent sheaf is in fact the notion of Cartesian presheaf, as he considers mostly the discrete topology. So his theorem on the equivalence of quasi-coherent sheaves and comodules over a Hopf algebroid is related to ours, but posed in a different framework. This point of view is also embraced by Goerss in [Go] who defines quasi-coherent sheaves trough the Cartesian condition.

We have to mention also Pribble's Ph. D. thesis [P]. We have used several times the exact sequence associated to the monad (or triple) corresponding to a presentation. This has allowed us to simplify some arguments.

In [SP], they put very few restrictions on the diagonal, which makes the situation more complicated. In the general setting a sheaf is quasi-coherent if, and only if, is Cartesian and locally quasi-coherent. As we have already remarked, they work over big sites for which functoriality is easy but sacrifices other useful features for the usual constructions in cohomology.

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1. Sheaves on ringed sites

1.0. In this paper we will use as underlying axiomatics that of von Neumann, Gödel and Bernays using sets and classes. A category C has a class of objects and a class of arrows but the class of maps between two objets $A, B \in \mathsf{C}$, denoted $\mathrm{Hom}_{\mathsf{C}}(A,B)$, is always assumed to be a set. For readers fond of Bourbaki-Grothendieck universes (and for comparison with results in [SGA 4_1]) fix two universes $\mathbb{U} \in \mathbb{V}$ and identify elements of $\mathbb{U}$ with sets and elements of $\mathbb{V}$ with classes.

1.1. Ringed categories. Let C be a small[2] category. We will denote by $\mathsf{Pre}(\mathsf{C}) := \mathsf{Fun}(\mathsf{C}^{\circ}, \mathsf{Ab})$ the category of contravariant functors from C to abelian groups. We will usually refer to objects in this category as *presheaves on* C. We will say that $(\mathsf{C}, \mathscr{O})$ is a ringed category if $\mathscr{O}$ is a ring object in $\mathsf{Pre}(\mathsf{C})$. In other words $\mathscr{O} \colon \mathsf{C}^{\circ} \to \mathsf{Ring}$ is a presheaf, or else $\mathscr{O}$ is a presheaf together with a couple of maps $\cdot \colon \mathscr{O} \otimes \mathscr{O} \to \mathscr{O}$ and $1 \colon \mathbb{Z}_{\mathsf{C}} \to \mathscr{O}$, with $\mathbb{Z}_{\mathsf{C}}$ denoting the constant presheaf with value $\mathbb{Z}$; this data makes commutative the usual diagrams expressing the associativity and commutativity of the product $\cdot$ and the fact that 1 is a unit.

1.2. Modules. Let $(\mathsf{C}, \mathscr{O})$ be a ringed category we will denote by $\mathsf{Pre}(\mathsf{C}, \mathscr{O})\text{-}\mathsf{Mod}$ or simply by $\mathsf{Pre}\text{-}\mathscr{O}\text{-}\mathsf{Mod}$, if no confusion arises, the category of presheaves $\mathscr{M} \in \mathsf{Pre}(\mathsf{C})$ equipped with an action of $\mathscr{O}$, $\mathscr{O} \otimes \mathscr{M} \to \mathscr{M}$, that makes commutative the diagrams expressing the fact that the sections of $\mathscr{M}$ are modules over the sections of $\mathscr{O}$ and the restriction maps are linear. We will say that $\mathscr{M}$ is an $\mathscr{O}$-Module.

[2] Its class of objects is a set, not a proper class.

1.3. Cartesian presheaves. Let $(\mathsf{C},\mathscr{O})$ be a ringed category. We will say that an $\mathscr{O}$-module $\mathscr{M}$ is a *Cartesian* presheaf if for every morphism $f : X \to Y$ in C the $\mathscr{O}(X)$-linear map

$$\mathscr{M}(f)^{\mathsf{a}} : \mathscr{O}(X) \otimes_{\mathscr{O}(Y)} \mathscr{M}(Y) \longrightarrow \mathscr{M}(X)$$

adjoint to the $\mathscr{O}(Y)$-linear map $\mathscr{M}(f) : \mathscr{M}(Y) \to \mathscr{M}(X)$ is an isomorphism. We will denote the category of Cartesian presheaves on $(\mathsf{C},\mathscr{O})$ by $\mathsf{Crt}(\mathsf{C},\mathscr{O})$.

1.4. Ringed sites and topos. As usual we will call a small category C a *site* if it is equipped with a Grothendieck topology τ [SGA 4_1, Exp. II §1]. If it is necessary to make it explicit, we will denote the site by (C,τ), otherwise we will keep the notation C. A site has associated a *topos*, its category of sheaves of sets. For a site (C,τ) we will denote its associated topos by C_τ. We may also consider sheaves with additional algebraic structures, in particular, a sheaf of rings $\mathscr{O}$. We will call the pair $(\mathsf{C},\mathscr{O})$ a *ringed site* and $(\mathsf{C}_\tau,\mathscr{O})$ a *ringed topos*. We will always refer with the notation $(\mathsf{C}_\tau,\mathscr{O})$-$\mathsf{Mod}$ or simply $\mathscr{O}$-Mod to the category of *sheaves* of modules over the sheaf of rings $\mathscr{O}$, and not to the biggest category of presheaves of modules.

1.5. Let C be a site and X and object in C. We will consider the category whose objects are morphisms $u : U \to X$, and morphisms are commutative triangles.[3] We will denote such an object by (U,u), or simply by U if no confusion arises. The category C/X is a site. The coverings of the corresponding topology of an object U are the families of morphisms $\{f_i : U_i \to U\}$ in C/X that are coverings in C. If $(\mathsf{C},\mathscr{O})$ is a ringed site, then $(\mathsf{C}/X,\mathscr{O}|_X)$ is a ringed site with the induced sheaf of rings $\mathscr{O}|_X$ given by $\mathscr{O}|_X(U,u) = \mathscr{O}(U)$. If $\mathscr{M}$ is a sheaf of $\mathscr{O}$-Modules on the site $(\mathsf{C},\mathscr{O})$ we define the sheaf restriction of $\mathscr{M}$ to C/X, denoted $\mathscr{M}|_X$, as the sheaf of $\mathscr{O}|_X$-Modules on the site $(\mathsf{C}/X,\mathscr{O}|_X)$ given by $\mathscr{M}|_X(U,u) = \mathscr{M}(U)$.

1.6. Quasi-coherent sheaves. Let $(\mathsf{C},\mathscr{O})$ be a ringed site with a topology τ. We will say that a sheaf of $\mathscr{O}$-Modules $\mathscr{M}$ is quasi-coherent if for every object $X \in \mathsf{C}$ there is a covering $\{p_i : X_i \to X\}_{i\in L}$ such that for the restriction of $\mathscr{M}$ to each C/X_i, there is a presentation

$$\mathscr{O}|_{X_i}^{(I)} \longrightarrow \mathscr{O}|_{X_i}^{(J)} \longrightarrow \mathscr{M}|_{X_i} \longrightarrow 0,$$

where by $\mathscr{F}^{(I)}$ we denote a coproduct of copies of a sheaf $\mathscr{F}$ indexed by some set I. If C is the site of open sets of a classical topology, this definition agrees with [EGA I, **0**, (5.1.3)]. We will denote the category of quasi-coherent sheaves on $(\mathsf{C},\mathscr{O})$ by $\mathsf{Qco}(\mathsf{C}_\tau,\mathscr{O})$.

1.7. Let $\mathsf{f} : (\mathsf{C},\tau) \to (\mathsf{D},\sigma)$ be a continuous functor of sites [SGA 4_1, III, (1.1)] such that f preserves fibered products. In view of [SGA 4_1, III, (1.6)], the continuity of f is equivalent to the fact that it takes coverings in (C,τ) to coverings in (D,σ).

[3] Sometimes called *the comma category*.

Whenever C possesses colimits, this morphism induces a pair of adjoint functors between the associated topos

$$\mathsf{D}_\sigma \underset{\mathsf{f}_*}{\overset{\mathsf{f}^{-1}}{\leftrightarrows}} \mathsf{C}_\tau .$$

Let us recall first the definition of f_*. If $\mathscr{G} \in \mathsf{D}_\sigma$ and $V \in \mathsf{C}$, then

$$(\mathsf{f}_*\mathscr{G})(V) = \mathscr{G}(\mathsf{f}(V)).$$

Let now $\mathscr{F} \in \mathsf{C}_\tau$, $\mathsf{f}^{-1}\mathscr{F}$ is defined as the sheaf associated to the presheaf that assigns to each $U \in \mathsf{D}$ the colimit

$$\varinjlim_{\mathbf{I}_U^{\mathsf{f}}} \mathscr{F}(V),$$

where $\mathbf{I}_U^{\mathsf{f}}$ is the category whose objects are the pairs (V,g) with $V \in \mathsf{C}$ and $g\colon U \to \mathsf{f}(V)$ is a morphism in D, and whose morphisms

$$h\colon (V,g) \to (V',g')$$

are defined as morphisms

$$h\colon V \to V' \text{ such that } \mathsf{f}(h)g = g'$$

(see [SGA 4_1, I, 5.1] and [SGA 4_1, III, 1.2.1] where f^{-1} is denoted f^s). Notice that, in general, f^{-1} might not be exact. If f^{-1} is exact, we say that the pair $(\mathsf{f}_*,\mathsf{f}^{-1})$ is a morphism of topos from D_σ to C_τ according to [SGA 4_1, IV 3.1].

We want to extend this formalism to the case of ringed sites. To have the possibility of transporting algebraic structures through the inverse image functor, we need the following lemma, communicated to us by Olsson.

Lemma 1.8. *Let $\mathbf{I}$ be a category with finite products and let $F_1,\dots,F_r$ a finite family of functors $F_i\colon \mathbf{I}^\circ \to \mathsf{Set}$, $i \in \{1\dots r\}$. The natural map*

$$\varinjlim_{\mathbf{I}} (F_1 \times \cdots \times F_r) \longrightarrow \left(\varinjlim_{\mathbf{I}} F_1\right) \times \cdots \times \left(\varinjlim_{\mathbf{I}} F_r\right)$$

is an isomorphism.

Proof. By using induction, we may assume that $r = 2$. Notice that

$$\varinjlim_{\mathbf{I}} F_1 \times \varinjlim_{\mathbf{I}} F_2$$

is identical to

$$\varinjlim_{\mathbf{I}\times\mathbf{I}} F_1 \overline{\times} F_2$$

where $F_1 \overline{\times} F_2\colon \mathbf{I}^\circ \times \mathbf{I}^\circ \to \mathsf{Set}$ is the functor that sends a pair $(U,V) \in \mathbf{I}^\circ \times \mathbf{I}^\circ$ to $F_1(U) \times F_2(V)$. The diagonal functor $\Delta\colon \mathbf{I}^\circ \to \mathbf{I}^\circ \times \mathbf{I}^\circ$ induces a map

$$z\colon \varinjlim_{\mathbf{I}} F_1 \times F_2 \longrightarrow \varinjlim_{\mathbf{I}\times\mathbf{I}} F_1 \overline{\times} F_2 = \varinjlim_{\mathbf{I}} F_1 \times \varinjlim_{\mathbf{I}} F_2$$

We will check first that z is surjective using the fact that $\mathbf{I}$ has products. Indeed, for any two objects $U,V \in \mathbf{I}$ and $u \in F_1(U)$, $v \in F_2(V)$ the element

$$[(u,v)] \in \varinjlim_{\mathbf{I}\times\mathbf{I}} F_1 \overline{\times} F_2$$

equals the class of $(F_1(p_1)(u), F_2(p_2)(v)) \in F_1(U\times V)\times F_2(U\times V)$ via the natural maps $p_1 \colon U\times V \to U$, $p_2 \colon U\times V \to V$, so any element lies in the image of z.

Now, we prove that z is injective. Consider the functor $\pi\colon \mathbf{I}^\circ \times \mathbf{I}^\circ \to \mathbf{I}^\circ$ that sends (U,V) to $U\times V$ (notice that the product is taken in $\mathbf{I}$). It gives the following natural transformation

$$F_1 \overline{\times} F_2 \longrightarrow (F_1 \times F_2)\circ \pi$$

defined as

$$F_1(p_1)\times F_2(p_2)\colon F_1(U)\times F_2(V) \longrightarrow F_1(U\times V)\times F_2(U\times V)$$

on $(U,V)\in \mathbf{I}^\circ \times \mathbf{I}^\circ$. The functor π induces a map

$$z'\colon \varinjlim_{\mathbf{I}\times\mathbf{I}} F_1 \overline{\times} F_2 \longrightarrow \varinjlim_{\mathbf{I}} F_1 \times F_2.$$

To see that z is injective is enough to check that $z'z = \mathrm{id}$. The map $z'z$ sends the class of $(u_1,u_2) \in F_1(U)\times F_2(U)$ to the class of $(F_1(p_1)(u_1), F_2(p_2)(u_2)) \in F_1(U\times U)\times F_2(U\times U)$. Let $\delta\colon U \to U\times U$ be the diagonal morphism. As we claimed, the class of $(F_1(p_1)(u_1), F_2(p_2)(u_2))$ equals the class of the element

$$(F_1(p_1\delta)(u_1), F_2(p_2\delta)(u_2)) = (u_1,u_2). \qquad \square$$

1.9. Let $\mathsf{f}\colon (\mathsf{C},\tau) \to (\mathsf{D},\sigma)$ be a continuous functor of sites as in 1.7. Let $\mathscr{O}$ be a sheaf of rings on (C,τ) and $\mathscr{P}$ a sheaf of rings on (D,σ). Let us be given a morphism of sheaves of rings $\mathsf{f}^\#\colon \mathscr{O} \to \mathsf{f}_*\mathscr{P}$ or, equivalently, its adjoint map $\mathsf{f}_\#\colon \mathsf{f}^{-1}\mathscr{O} \to \mathscr{P}$.

Consider first the case in which $(\mathsf{f}_*, \mathsf{f}^{-1})$ is a morphism of topos. In this case we say that $((\mathsf{f}_*, \mathsf{f}^{-1}), \mathsf{f}_\#)$ is a morphism of *ringed* topos [SGA 4_1, IV, 13.1] and provides an adjunction

$$\mathscr{P}\text{-}\mathsf{Mod} \underset{\mathsf{f}_*}{\overset{\mathsf{f}^*}{\leftrightarrows}} \mathscr{O}\text{-}\mathsf{Mod}.$$

The functor $\mathsf{f}^*\colon \mathscr{O}\text{-}\mathsf{Mod} \to \mathscr{P}\text{-}\mathsf{Mod}$ is defined by

$$\mathsf{f}^*\mathscr{F} := \mathscr{P} \otimes_{\mathsf{f}^{-1}\mathscr{O}} \mathsf{f}^{-1}\mathscr{F}$$

for a $\mathscr{O}$-module $\mathscr{F}$. Observe that for a $\mathscr{P}$-module $\mathscr{G}$, $\mathsf{f}_*\mathscr{G}$ inherits a structure of $\mathscr{O}$-module via $\mathsf{f}^\#$ and we have the claimed adjunction $\mathsf{f}^* \dashv \mathsf{f}_*$.

In general f^{-1} may not be exact, and the previous discussion does not apply. However, if we assume that the category $\mathbf{I}_U^{\mathsf{f}}$ has finite products, then by Lemma 1.8, the functor that defines the inverse image for presheaves commutes with finite products. The sheafification functor commutes also with finite products since it is exact. It follows that f^{-1} also commutes with finite products. Thus, $\mathsf{f}^{-1}\mathscr{O}$ has canonically the structure of a sheaf of rings and

also $\mathsf{f}_{\#}$ is a homomorphism. By the same reason, if $\mathscr{F}$ is an $\mathscr{O}$-module then $\mathsf{f}^{-1}\mathscr{F}$ is an $\mathsf{f}^{-1}\mathscr{O}$-module. The functor $\mathsf{f}^*\colon \mathscr{O}\text{-}\mathsf{Mod} \to \mathscr{P}\text{-}\mathsf{Mod}$ given by

$$\mathsf{f}^*\mathscr{F} := \mathscr{P} \otimes_{\mathsf{f}^{-1}\mathscr{O}} \mathsf{f}^{-1}\mathscr{F}$$

provides an adjunction $\mathsf{f}^* \dashv \mathsf{f}_*$ as in the previous situation.

Notice that we use here f^{-1} for general sheaves and reserve f^* for modules over ringed topos, contradicting the usage of [SGA 4_1, IV].

Remark. The perspicuous reader will notice that Lemma 1.8 is also implicitly used in [O, Def. 3.11].

2. Cartesian presheaves on affine schemes

2.1. Let $X = \mathrm{Spec}(A)$ be an affine scheme. We will work with the category Aff/X of flat finitely presented affine schemes over X, in other words the opposite category to the category of flat finite presentation A-algebras. Note that this is an essentially small category. We will endow it with the topology whose coverings are finite families $\{p_i : U_i \to U\}_{i\in L}$ such that p_i are flat finite presentation maps such that $\cup_{i\in L} \mathrm{Im}(p_i) = U$. We will denote this site by $\mathsf{Aff}_{\mathsf{fppf}}/X$ and by X_{fppf} its associated topos. Due to the fact that we limit the possible objects to those flat of finite presentation, this site should not be regarded as a big site and sometimes we will refer to it informally as the *small flat site*.

Let $\widetilde{A}$ or $\mathscr{O}_X$ denote the ring valued sheaf defined by $\widetilde{A}(\mathrm{Spec}(B)) = B$. Note that $\widetilde{A}$ is a sheaf on $\mathsf{Aff}_{\mathsf{fppf}}/X$ as follows from faithfully flat descent [SGA 4_2, VII, 2c], see also [SGA 1, VIII, 1.6] and [SGA 3_1, IV, 6.3.1]. We will denote the corresponding ringed topos as $(X_{\mathsf{fppf}}, \widetilde{A})$ or $(X_{\mathsf{fppf}}, \mathscr{O}_X)$.

Let $\mathsf{Crt}(X) := \mathsf{Crt}(\mathsf{Aff}/X, \widetilde{A})$. We recall the somewhat classical

Proposition 2.2. *With the previous notation, there is an equivalence of categories*

$$\mathsf{Crt}(X) \cong A\text{-}\mathsf{Mod}.$$

Proof. Consider the functors $\Gamma(X,-)\colon \mathsf{Crt}(X) \to A\text{-}\mathsf{Mod}$ and $(-)_{\mathsf{crt}}\colon A\text{-}\mathsf{Mod} \to \mathsf{Crt}(X)$ defined by $\Gamma(X,\mathscr{M}) := \mathscr{M}(X, \mathrm{id}_X)$ and $M_{\mathsf{crt}}(\mathrm{Spec}(R)) := R \otimes_A M$. It is clear that they are mutually quasi-inverse. □

Since each Cartesian presheaf in $\mathsf{Crt}(X)$ is isomorphic to a presheaf of the form M_{crt}, then it is a sheaf [SGA 4_2, VII, 2c]. The category $\mathsf{Crt}(X)$ is a full subcategory of the category of sheaves of $\widetilde{A}$-Modules closed under the formation of kernels, cokernels and coproducts, and it is an abelian category.

2.3. Let $f\colon X \to Y$ be a morphism of affine schemes with $X = \mathrm{Spec}(B)$ and $Y = \mathrm{Spec}(A)$. We have a continuous functor

$$f^{\mathsf{Aff}}\colon \mathsf{Aff}_{\mathsf{fppf}}/Y \longrightarrow \mathsf{Aff}_{\mathsf{fppf}}/X$$

given on objects by $f^{\mathsf{Aff}}(V,v) := (V \times_Y X, p_2)$ with p_2 denoting the projection and on maps by the functoriality of pull-backs. It is clear that f^{Aff} preserves fibered products, therefore by 1.7, induces a pair of adjoint functors

$$X_{\mathsf{fppf}} \underset{f_*}{\overset{f^{-1}}{\leftrightarrows}} Y_{\mathsf{fppf}},$$

where we are taking the usual shortcuts $f^{-1} := (f^{\mathsf{Aff}})^{-1}$ and $f_* := (f^{\mathsf{Aff}})_*$.

Moreover, we have the morphism of sheaves of rings

$$f^{\mathsf{Aff}\,\#} \colon \widetilde{A} \longrightarrow f_*\widetilde{B}$$

given by $f^{\mathsf{Aff}\,\#}(V,v) \colon R \to R \otimes_A B$, $f^{\mathsf{Aff}\,\#}(V,v)(r) = r \otimes 1$, for $V = \mathrm{Spec}(R)$. Since the category $\mathbf{I}^{f^{\mathsf{Aff}}}_{(U,u)}$ has finite products, by 1.9 we have the adjunction

$$\widetilde{B}\text{-}\mathsf{Mod} \underset{f_*}{\overset{f^*}{\leftrightarrows}} \widetilde{A}\text{-}\mathsf{Mod}$$

where $f^* := (f^{\mathsf{Aff}})^*$. Notice that if f is a flat finitely presented morphism, then $f^*\mathscr{F}(U,u) = \mathscr{F}(U, fu)$.

Remark. We will abuse notation slightly and write $\mathscr{G}(U)$ for $\mathscr{G}(U,u)$ for any $(U,u) \in \mathsf{Aff}/X$ and $\mathscr{G} \in X_{\mathsf{fppf}}$, if the morphism u is understood.

Proposition 2.4. *It holds that*

(*i*) *If* $\mathscr{F} \in \mathsf{Crt}(Y)$ *then* $f^*\mathscr{F} \in \mathsf{Crt}(X)$.
(*ii*) *If* $\mathscr{G} \in \mathsf{Crt}(X)$ *then* $f_*\mathscr{G} \in \mathsf{Crt}(Y)$.

Proof. (*i*) Let $(U,u) \in \mathsf{Aff}_{\mathsf{fppf}}/X$ with $U = \mathrm{Spec}(S)$. Denote $\mathbf{I}^{f^{\mathsf{Aff}}}_{(U,u)}$ by $\mathbf{I}_U$ and let

$$A_U := \varinjlim_{\mathbf{I}_U} \widetilde{A}(V).$$

The sheaf $f^*\mathscr{F}$ is the sheaf associated to the presheaf P given by

$$P(U,u) = S \otimes_{A_U} \varinjlim_{\mathbf{I}_U} \mathscr{F}(V).$$

We have for every S-module N the following isomorphisms

$$\begin{aligned}
\mathrm{Hom}_S(S \otimes_{A_U} \varinjlim_{\mathbf{I}_U} \mathscr{F}(V), N) &\simeq \mathrm{Hom}_{A_U}(\varinjlim_{\mathbf{I}_U} \mathscr{F}(V), N)\\
&\simeq \varprojlim_{\mathbf{I}_U} \mathrm{Hom}_{\widetilde{A}(V)}(\mathscr{F}(V), N)\\
&\simeq \varprojlim_{\mathbf{I}_U} \mathrm{Hom}_S(S \otimes_{\widetilde{A}(V)} \mathscr{F}(V), N)\\
&\simeq \mathrm{Hom}_S(\varinjlim_{\mathbf{I}_U} (S \otimes_{\widetilde{A}(V)} \mathscr{F}(V)), N).
\end{aligned}$$

By Yoneda's lemma, it follows that $f^*\mathscr{F}$ is isomorphic to the sheaf associated to the presheaf P' such that

$$P'(U,u) = \varinjlim_{\mathbf{I}_U} S \otimes_{\widetilde{A}(V)} \mathscr{F}(V).$$

Let $h\colon ((V,v),g) \to ((V',v'),g')$ be a morphism in $\mathbf{I}_U$, with $V = \operatorname{Spec}(R)$, $V' = \operatorname{Spec}(R')$ and $h = \operatorname{Spec}(\varphi)$. We have the commutative diagram

$$
\begin{tikzcd}
S \otimes_{R'} \mathscr{F}(V') \arrow[r, "\mathrm{id}\otimes\mathscr{F}(h)"] & S \otimes_R \mathscr{F}(V) \\
S \otimes_{R'} R' \otimes_A \mathscr{F}(Y) \arrow[r, "\mathrm{id}\otimes\varphi\otimes\mathrm{id}"] \arrow[u, "\mathrm{id}\otimes\mathscr{F}(v')^{\mathsf{a}}"] \arrow[d, equal] & S \otimes_R R \otimes_A \mathscr{F}(Y) \arrow[u, "\mathrm{id}\otimes\mathscr{F}(v)^{\mathsf{a}}"'] \arrow[d, equal] \\
S \otimes_A \mathscr{F}(Y) \arrow[r, equal] & S \otimes_A \mathscr{F}(Y)
\end{tikzcd}
$$

where the top horizontal map is the transition morphism and the vertical maps are isomorphisms by the definition of Cartesian presheaf and the properties of the tensor product. Thus, $P'(U,u) = S \otimes_A \mathscr{F}(Y)$ and therefore $f^*\mathscr{F}$ is isomorphic to $(B \otimes_A \mathscr{F}(Y))_{\mathsf{crt}}$.

(*ii*) Let $h\colon (V,v) \to (V',v')$ be a morphism in $\mathsf{Aff}_{\mathsf{fppf}}/Y$, $V = \operatorname{Spec}(R)$, $V' = \operatorname{Spec}(R')$ and $h = \operatorname{Spec}(\varphi)$. We have to prove that the morphism

$$f_*\mathscr{G}(h)^{\mathsf{a}}\colon R \otimes_{R'} f_*\mathscr{G}(V') \longrightarrow f_*\mathscr{G}(V)$$

is an isomorphism (see 1.3). We have that $f_*\mathscr{G}(V) = \mathscr{G}(V \times_Y X)$ and similarly for $f_*\mathscr{G}(V')$. Notice that $f_*\mathscr{G}(h) = \mathscr{G}(h \times_Y \mathrm{id}_X)$, thus the morphism

$$\mathscr{G}(h \times_Y \mathrm{id}_X)^{\mathsf{a}}\colon (R \otimes_A B) \otimes_{R' \otimes_A B} \mathscr{G}(V' \times_Y X) \longrightarrow \mathscr{G}(V \times_Y X)$$

is an isomorphism because $\mathscr{G}$ is Cartesian. The result follows because $f_*\mathscr{G}(h)^{\mathsf{a}}$ is identified with the composition of the following isomorphisms

$$R \otimes_{R'} \mathscr{G}(V' \times_Y X) \xrightarrow{\sim} (R \otimes_A B) \otimes_{R' \otimes_A B} \mathscr{G}(V' \times_Y X) \xrightarrow{\mathscr{G}(h \times \mathrm{id})^{\mathsf{a}}} \mathscr{G}(V \times_Y X).$$

Notice that $f_*(\mathscr{G})$ is isomorphic to $({}_A\mathscr{G}(X))_{\mathsf{crt}}$, where ${}_A\mathscr{G}(X)$ means $\mathscr{G}(X)$ considered as an A-module. □

Corollary 2.5. *A morphism of affine schemes $f\colon X \to Y$ induces a pair of adjoint functors*

$$\mathsf{Crt}(X) \underset{f_*}{\overset{f^*}{\leftrightarrows}} \mathsf{Crt}(Y).$$

Remark. We leave as an exercise for the interested reader to recognize that the previous adjunction agrees with the classical one

$$\mathsf{Qco}(X_{\mathsf{fppf}}, \mathscr{O}_X) \underset{f_*}{\overset{f^*}{\leftrightarrows}} \mathsf{Qco}(Y_{\mathsf{fppf}}, \mathscr{O}_Y),$$

via the fact that in both cases the (pre)sheaves can be represented by its module of global sections. Notice that we consider the fppf topology instead of the usual Zariski topology but this does not change essentially the category of quasi-coherent sheaves. We will see later (in Theorem 3.11) the agreement of quasi-coherent sheaves with Cartesian presheaves in a much more general setting that includes all quasi-compact and semi-separated schemes.

3. Quasi-coherent sheaves on geometric stacks

3.1. We will follow the conventions of [LMB], specially for the definition of *algebraic* stack. But for our purposes we will restrict to what we call *geometric stacks* after Lurie (see [L]). So, from now on, a geometric stack X will be a stack on the étale topology of (affine) schemes over a base scheme (that will not play any role in what follows)[4] such that:

(*i*) the stack X is semi-separated, i.e. the diagonal morphism $\delta\colon \mathsf{X} \to \mathsf{X} \times \mathsf{X}$ is representable by *affine* schemes;

(*ii*) the stack X is algebraic and quasi-compact, this amounts to the existence of a smooth and surjective morphism $p\colon X \to \mathsf{X}$ from an *affine* scheme X.

We will refer to the map $p\colon X \to \mathsf{X}$ as a *presentation* of X.

A map representable by affine schemes is usually denominated an affine morphism [LMB, 3.10]. Observe that every morphism $u\colon U \to \mathsf{X}$ from an affine scheme U is affine. Indeed, if $v\colon V \to \mathsf{X}$ is another morphism with V an affine scheme, we have that $U \times_{\mathsf{X}} V \simeq \mathsf{X} \times_{\mathsf{X}\times\mathsf{X}} (U \times V)$ is 1-isomorphic to an affine scheme and this 1-isomorphism is an isomorphism because $U \times_{\mathsf{X}} V$ is in fact a category fibered in sets. We will identify it with the spectrum of its global sections. We recall that the pair $(X, X \times_{\mathsf{X}} X)$ with the obvious structure morphisms is a scheme in groupoids whose associated stack is X, *i.e.*

$$\mathsf{X} = [(X, X \times_{\mathsf{X}} X)]$$

with the notation as in [LMB, (3.4.3)].

Lemma 3.2. *Let $f\colon \mathsf{X} \to \mathsf{Y}$ a morphism of geometric stacks. The relative diagonal $\delta_{\mathsf{X}/\mathsf{Y}}\colon \mathsf{X} \to \mathsf{X} \times_{\mathsf{Y}} \mathsf{X}$ is an affine morphism.*

Proof. Consider the following composition

$$\mathsf{X} \xrightarrow{\Gamma_f} \mathsf{X} \times \mathsf{Y} \xrightarrow{p_2} \mathsf{Y}.$$

Notice that $f \simeq p_2 \circ \Gamma_f$. Semi-separatedness on schemes is preserved by base change [AJPV, Proposition 2.2 (iii)], so by [LMB, Remarque (4.14.1)] it is also preserved by base change on algebraic stacks, this guarantees the semi-separatedness of p_2. The semi-separatedness of Γ_f follows from the fact that is a quasi-section, so its diagonal map is an equivalence. □

Proposition 3.3. *Let*

$$\begin{tikzcd} \mathsf{X}' \arrow[r, "f'"] \arrow[d, "g'"'] & \mathsf{Y}' \arrow[d, "g"] \\ \mathsf{X} \arrow[r, "f"'] \arrow[ur, Rightarrow, "\phi"] & \mathsf{Y} \end{tikzcd}$$

be a 2-Cartesian square of algebraic stacks. If X, Y and Y' are geometric stacks, then so is X'.

[4] In fact, it can be shown that the fppf topology yields the same category of stacks.

Proof. First, $\mathbf{X}'$ is an algebraic stack by [LMB, (4.5)(i)].

Let us check that the diagonal morphism $\delta_{\mathbf{X}'} : \mathbf{X}' \to \mathbf{X}' \times \mathbf{X}'$ is affine. Factor it as

$$\mathbf{X}' \xrightarrow{\delta_{\mathbf{X}'/\mathbf{X}}} \mathbf{X}' \times_{\mathbf{X}} \mathbf{X}' \xrightarrow{h} \mathbf{X}' \times \mathbf{X}'$$

where h is the morphism in the following 2-Cartesian square

$$\begin{tikzcd}
\mathbf{X}' \times_{\mathbf{X}} \mathbf{X}' \arrow[r, "h"] \arrow[d, "g''"'] & \mathbf{X}' \times \mathbf{X}' \arrow[d, "g' \times g'"] \\
\mathbf{X} \arrow[r, "\delta_{\mathbf{X}}"] \arrow[ur, Rightarrow, "\phi'"] & \mathbf{X} \times \mathbf{X}
\end{tikzcd}$$

The morphism h is affine because it is obtained by base change from $\delta_{\mathbf{X}}$. The morphism $\delta_{\mathbf{X}'/\mathbf{X}} \simeq \delta_{\mathbf{Y}'/\mathbf{Y}} \times_{\mathbf{X}} \mathbf{X}'$, so by Lemma 3.2, it is affine. It follows that $\delta_{\mathbf{X}'}$ is affine as wanted.

Finally, given presentations $p : X \to \mathbf{X}$ and $q : Y' \to \mathbf{Y}'$ we have an induced morphism

$$p \times_{\mathbf{Y}} q : X \times_{\mathbf{Y}} Y' \longrightarrow \mathbf{X}'$$

Notice that $X \times_{\mathbf{Y}} Y'$ is affine over X because is the base change of the map q that is affine, therefore $X \times_{\mathbf{Y}} Y'$ is an affine scheme. The morphism $p \times_{\mathbf{Y}} q$ is smooth and surjective because both conditions are stable by base change applying [LMB, Prop. (4.13) et Rem. (4.14.1)] in view of [EGA I, (1.3.9)]. This makes of $p \times_{\mathbf{Y}} q$ a presentation and the proof is complete. □

3.4. For $\mathbf{X}$ a geometric stack, we define the category $\mathsf{Aff}/\mathbf{X}$ as follows. Objects are pairs (U, u) with U an affine scheme and $u : U \to \mathbf{X}$ a flat finitely presented 1-morphism of stacks. Morphisms $(f, \alpha) : (U, u) \to (V, v)$ are commutative diagrams of 1-morphisms of stacks. Let us spell this out. To give such an (f, α) amounts to give a diagram

$$\begin{tikzcd}
U \arrow[rr, "f"] \arrow[dr, "u"'] & \arrow[d, phantom, "\overset{\alpha}{\Rightarrow}"] & V \arrow[dl, "v"] \\
& \mathbf{X} &
\end{tikzcd}$$

where $f : U \to V$ is a morphism of affine schemes making the diagram 2-commutative, through the 2-cell $\alpha : u \Rightarrow vf$. The composition of morphisms is given by

$$(g, \beta) \circ (f, \alpha) = (gf, \beta f \circ \alpha).$$

Notice that $\mathsf{Aff}/\mathbf{X}$ is an essentially small category. In fact, let $(U, u) \in \mathsf{Aff}/\mathbf{X}$ with $U = \operatorname{Spec}(B)$, by the semi-separateness of $\mathbf{X}$, the scheme $U \times_{\mathbf{X}} X$ is affine. This affine scheme is isomorphic to the spectrum of a finitely presented algebra B' over the ring of global sections of the structure sheaf of X. By faithful flatness B is isomorphic to a subring of B'. This makes sense of what follows.

3.5. The category $\mathsf{Aff}/\mathbf{X}$ is ringed by the presheaf $\mathscr{O}\colon \mathsf{Aff}/\mathbf{X} \to \mathsf{Ring}$, defined by $\mathscr{O}(U,u) = B$ with $U = \operatorname{Spec}(B)$. We define Cartesian presheaves on a geometric stack $\mathbf{X}$ as Cartesian presheaves over this ringed category,

$$\mathsf{Crt}(\mathbf{X}) := \mathsf{Crt}(\mathsf{Aff}/\mathbf{X}, \mathscr{O}).$$

Now, on $\mathsf{Aff}/\mathbf{X}$ we may define a topology by declaring as covering the finite families $\{(f_i,\alpha_i)\colon (U_i,u_i) \to (V,v)\}_{i\in I}$, with every f_i a flat finitely presented map, that are jointly surjective. It becomes a site that we denote $\mathsf{Aff}_{\mathsf{fppf}}/\mathbf{X}$. The associated topos, *i.e.* the category of sheaves of sets on $\mathsf{Aff}_{\mathsf{fppf}}/\mathbf{X}$ is denoted simply $\mathbf{X}_{\mathsf{fppf}}$. The site $\mathsf{Aff}_{\mathsf{fppf}}/\mathbf{X}$ is a ringed site by the sheaf of rings associated to the presheaf $\mathscr{O}$ that we keep denoting the same.[5] We define

$$\mathsf{Qco}(\mathbf{X}) := \mathsf{Qco}(\mathbf{X}_{\mathsf{fppf}}, \mathscr{O}).$$

Remark.

(i) Every smooth morphism is flat of finite presentation, therefore our site is finer than the usual *lisse-étale* topology, see [LMB, (12.1)].

(ii) The topos $\mathbf{X}_{\mathsf{fppf}}$ has enough points. The argument is the same as for the lisse-étale topos, see [LMB, (12.2.2)].

In what follows we will abuse notation slightly and write $\mathscr{F}(U)$ for $\mathscr{F}(U,u)$ and analogously for any object in $\mathsf{Aff}/\mathbf{X}$ if the morphism $u\colon U \to \mathbf{X}$ is obvious.

Lemma 3.6. *A Cartesian presheaf $\mathscr{F} \in \mathsf{Crt}(\mathbf{X})$ is actually a* sheaf *of $\mathscr{O}$-Modules.*

Proof. Let $(V,v) \in \mathsf{Aff}/\mathbf{X}$ with $V = \operatorname{Spec}(C)$. Let $\{(f_i,\alpha_i)\colon (U_i,u_i) \to (V,v)\}_{i\in I}$ be a covering with $U_i = \operatorname{Spec}(B_i)$. Let p_1^{ij} and p_2^{ij} be the two canonical projections from $U_i \times_V U_j$ to U_i and U_j, respectively. Consider $(U_i \times_V U_j, vf_ip_1^{ij}) \in \mathsf{Aff}/\mathbf{X}$ (notice that $vf_ip_1^{ij} = vf_jp_2^{ij}$). We have the diagram

$$\mathscr{F}(V) \xrightarrow{\ l\ } \textstyle\prod_{i\in I} \mathscr{F}(U_i) \overset{\rho_2}{\underset{\rho_1}{\rightrightarrows}} \prod_{i,j\in I} \mathscr{F}(U_i \times_V U_j)$$

where $l = (\mathscr{F}(f_i,\alpha_i))_{i\in I}$, for $(x_i) \in \prod_{i\in I} \mathscr{F}(U_i)$, we denote by $\rho_1(x_i)$ the element whose component in $\mathscr{F}(U_i \times_V U_j)$ is $\mathscr{F}(p_1^{ij}, \alpha_i^{-1}p_1^{ij})(x_i)$ and by $\rho_2(x_i)$ the element whose component in $\mathscr{F}(U_i \times_V U_j)$ is $\mathscr{F}(p_2^{ij}, \alpha_j^{-1}p_2^{ij})(x_j)$. Let us check that this diagram is an equalizer. Being $\mathscr{F}$ Cartesian, this diagram is isomorphic to the diagram

$$\mathscr{F}(V) \xrightarrow{\ l'\ } \textstyle\prod_{i\in I} B_i \otimes_C \mathscr{F}(V) \overset{\rho'_2}{\underset{\rho'_1}{\rightrightarrows}} \prod_{i,j\in I} (B_i \otimes_C B_j) \otimes_C \mathscr{F}(V)$$

where $l'(x) = (1 \otimes x)_{i\in I}$, for $x \in \mathscr{F}(V)$, and for $(b_i \otimes y_i)_{i\in I} \in \prod B_i \otimes_C \mathscr{F}(V)$, the elements $\rho'_1((b_i \otimes y_i)_{i\in I})$ and $\rho'_2((b_i \otimes y_i)_{i\in I})$ are those whose components in $(B_i \otimes_C B_j) \otimes_C \mathscr{F}(V)$ are $b_i \otimes 1 \otimes y_i$, and $1 \otimes b_j \otimes y_j$, respectively. Let $B := \prod_{i\in I} B_i$

[5] The next lemma will show that there is no ambiguity.

and the morphism $\varphi\colon C \to B$ induced by the covering $\{f_i : U_i \to V\}$. Since I is finite, it suffices to prove that the diagram

$$\mathscr{F}(V) \xrightarrow{l''} B \otimes_C \mathscr{F}(V) \underset{j_1 \otimes \mathrm{id}}{\overset{j_2 \otimes \mathrm{id}}{\rightrightarrows}} (B \otimes_C B) \otimes_C \mathscr{F}(V)$$

is an equalizer, where $l''(x) = 1 \otimes x$, $j_1(b) = b \otimes 1$ and $j_2(b) = 1 \otimes b$, for $x \in \mathscr{F}(V)$ and $b \in B$. As B is faithfully flat over C, the diagram

$$C \xrightarrow{\varphi} B \underset{j_1}{\overset{j_2}{\rightrightarrows}} B \otimes_C B$$

is an equalizer and stays so after tensoring by $\mathscr{F}(V)$ [SGA1, VIII 1.5]. The result follows. $\square$

Remark.

(*i*) The proof works for any topology coarser than fpqc, we have chosen the fppf topology because it gives a reasonable small site.

(*ii*) The category $\mathsf{Crt}(\mathbf{X})$ is a priori a full subcategory of $\mathsf{Pre}\text{-}\mathscr{O}_{\mathbf{X}}\text{-}\mathsf{Mod}$. The previous lemma implies that it is a full subcategory of $\mathscr{O}_{\mathbf{X}}\text{-}\mathsf{Mod}$, as $\mathsf{Qco}(\mathbf{X})$ is.

Proposition 3.7. *The category* $\mathsf{Crt}(\mathbf{X})$ *is a full subcategory of* $\mathscr{O}_{\mathbf{X}}\text{-}\mathsf{Mod}$ *closed under the formation of kernel, cokernels and coproducts, and therefore it is an abelian category.*

Proof. Let $\mathscr{K}$ be the kernel of the morphism $\mathscr{M} \to \mathscr{M}'$ with $\mathscr{M}, \mathscr{M}' \in \mathsf{Crt}(\mathbf{X})$ and $(f, \alpha)\colon (U, u) \to (V, v)$ be a morphism in $\mathsf{Aff}/\mathbf{X}$ with $U = \mathrm{Spec}(B)$ and $V = \mathrm{Spec}(C)$. If f is flat, we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & B \otimes_C \mathscr{K}(V) \arrow[r] \arrow[d] & B \otimes_C \mathscr{M}(V) \arrow[r] \arrow[d, "\wr"'] & B \otimes_C \mathscr{M}'(V) \arrow[d, "\wr"'] \\
0 \arrow[r] & \mathscr{K}(U) \arrow[r] & \mathscr{M}(U) \arrow[r] & \mathscr{M}'(U)
\end{tikzcd}$$

where the rows are exact and the midle and right vertical arrows are isomorphisms because $\mathscr{M}$ and $\mathscr{M}'$ are Cartesian sheaves. It follows that the remaining vertical arrow $\mathscr{K}(f, \alpha)^{\mathsf{a}} : B \otimes_C \mathscr{K}(V) \longrightarrow \mathscr{K}(U)$ is an isomorphism.

If f is not flat, let $p\colon X \to \mathbf{X}$ be an affine presentation with $X = \mathrm{Spec}\, A_0$. If p_1 and p'_1, and p_2 and p'_2 are the projections from $U \times_{\mathbf{X}} X$ and $V \times_{\mathbf{X}} X$ to U and V, and X, respectively, and $\beta\colon up_1 \Rightarrow pp_2$ and $\beta'\colon vp'_1 \Rightarrow pp'_2$ are the canonical 2-morphisms, we have a commutative diagram of affine schemes

$$\begin{tikzcd}
 & U \times_{\mathbf{X}} X \arrow[r, "p_1"] \arrow[d, "f'"] \arrow[dl, "p_2"'] & U \arrow[d, "f"] \\
X & V \times_{\mathbf{X}} X \arrow[l, "p'_2"] \arrow[r, "p'_1"'] & V
\end{tikzcd}$$

where the projections p_1, p_2, p'_1 and p'_2 are flat maps and $\beta' f' = \alpha p_1 \circ \beta$. Put $\mathrm{Spec}(B') = U \times_{\mathbf{X}} X$ and $\mathrm{Spec}(C') = V \times_{\mathbf{X}} X$. Since the projection p_1 is faithfully flat, to prove that the morphism $\mathscr{K}(f,\alpha)^{\mathsf{a}}$ is an isomorphism it suffices to verify that the morphism

$$\mathrm{id}_{B'} \otimes_B \mathscr{K}(f,\alpha)^{\mathsf{a}} \colon B' \otimes_B (B \otimes_C \mathscr{K}(V)) \longrightarrow B' \otimes_B \mathscr{K}(U)$$

is an isomorphism. But this morphism is the composition of the following chain of isomorphisms

$$\begin{aligned}
B' \otimes_B (B \otimes_C \mathscr{K}(V)) &\simeq B' \otimes_{C'} C' \otimes_C \mathscr{K}(V) && \\
&\simeq B' \otimes_{C'} \mathscr{K}(V \times_{\mathbf{X}} X, p p'_2) && (\text{via } \mathscr{K}(p'_1,(\beta')^{-1})) \\
&\simeq B' \otimes_{C'} (C' \otimes_{A_0} \mathscr{K}(X)) && (\text{via } \mathscr{K}(p'_2,\mathrm{id})) \\
&\simeq B' \otimes_{A_0} \mathscr{K}(X) && \\
&\simeq \mathscr{K}(U \times_{\mathbf{X}} X, p p_2) && (\text{via } \mathscr{K}(p_2,\mathrm{id})) \\
&\simeq B' \otimes_B \mathscr{K}(U) && (\text{via } \mathscr{K}(p_1,\beta^{-1}))
\end{aligned}$$

The presheaf cokernel of $\mathscr{M} \to \mathscr{M}'$ is a Cartesian presheaf and then it is the cokernel in the category of sheaves of $\mathscr{O}$-modules. Similarly, the coproduct of a set of Cartesian presheaves is a Cartesian presheaf and then it is the coproduct in the category of sheaves of $\mathscr{O}$-modules. □

Let $\mathbf{X}$ be a geometric stack and $p \colon X \to \mathbf{X}$ a presentation with X an affine scheme. Then p induces a continuous morphism of sites

$$p^{\mathsf{Aff}} \colon \mathsf{Aff}_{\mathsf{fppf}}/\mathbf{X} \longrightarrow \mathsf{Aff}_{\mathsf{fppf}}/X$$

that is given on objects by $p^{\mathsf{Aff}}(U,u) := (U \times_{\mathbf{X}} X, p_2)$, with $p_2 \colon U \times_{\mathbf{X}} X \to X$ the projection, and on maps by the functoriality of pull-backs.

As discussed in 1.7 this morphism induces a pair of adjoint functors

$$X_{\mathsf{fppf}} \underset{p_*}{\overset{p^{-1}}{\rightleftarrows}} \mathbf{X}_{\mathsf{fppf}}.$$

Notice that p is a flat finite presentation 1-morphism, so $(p^{-1}\mathscr{F})(V,v) = \mathscr{F}(V,pv)$, for $\mathscr{F} \in \mathbf{X}_{\mathsf{fppf}}$. In particular, $p^{-1}\mathscr{O}_{\mathbf{X}} = \mathscr{O}_X$, therefore p^{-1} induces a functor between the categories of sheaves of $\mathscr{O}$-Modules, *i.e.* p^{-1} agrees with the functor usually denoted p^*. So we have an adjunction as in 1.9

$$\mathscr{O}_X\text{-}\mathsf{Mod} \underset{p_*}{\overset{p^*}{\rightleftarrows}} \mathscr{O}_{\mathbf{X}}\text{-}\mathsf{Mod}, \tag{3.7.1}$$

with a particularly simple description of the functors involved, due to the fact that p induces a restriction functor between the corresponding sites.

Proposition 3.8. *It holds that*

(*i*) *If* $\mathscr{F} \in \mathsf{Crt}(\mathbf{X})$ *then* $p^*\mathscr{F} \in \mathsf{Crt}(X)$.
(*ii*) *If* $\mathscr{G} \in \mathsf{Crt}(X)$ *then* $p_*\mathscr{G} \in \mathsf{Crt}(\mathbf{X})$.

Proof. We prove (*i*). Let $h\colon (U,u) \to (V,v)$ be a morphism in $\mathsf{Aff}_{\mathsf{fppf}}/X$ with $U = \operatorname{Spec}(B)$ and $V = \operatorname{Spec}(C)$. We have to check that the morphism

$$(p^*\mathscr{F}(h))^{\mathsf{a}}\colon B \otimes_C p^*\mathscr{F}(V,v) \longrightarrow p^*\mathscr{F}(U,u)$$

is an isomorphism. But $p^*\mathscr{F}(V,v) = \mathscr{F}(V,pv)$ and analogously for (U,u). Therefore the previous morphism becomes

$$\mathscr{F}(h,\mathsf{id}_{pu})^{\mathsf{a}}\colon B \otimes_C \mathscr{F}(V,pv) \longrightarrow \mathscr{F}(U,pu)$$

where $(h,\mathsf{id}_{pu})\colon (U,pu) \to (V,pv)$ is a morphism in $\mathsf{Aff}_{\mathsf{fppf}}/\mathbf{X}$. But this map is an isomorphism because $\mathscr{F}$ is Cartesian.

The proof of (*ii*) is similar to the proof given in Proposition 2.4 (*ii*). $\square$

Corollary 3.9. *In this setting, the adjunction (3.7.1) restricts to*

$$\mathsf{Crt}(X) \underset{p_*}{\overset{p^*}{\rightleftarrows}} \mathsf{Crt}(\mathbf{X}).$$

Let us now prove a converse to Proposition 3.8 (*i*).

Lemma 3.10. *Let $\mathscr{F}$ be an $\mathscr{O}_{\mathbf{X}}$-module. If $p^*\mathscr{F} \in \mathsf{Crt}(X)$, then $\mathscr{F} \in \mathsf{Crt}(\mathbf{X})$.*

Proof. Set $\mathsf{T} := p_*p^*$. Since $\mathscr{F}$ is a sheaf, we have the equalizer of $\mathscr{O}_{\mathbf{X}}$-modules

$$\mathscr{F} \xrightarrow{\eta_{\mathscr{F}}} \mathsf{T}\mathscr{F} \underset{\mathsf{T}\eta_{\mathscr{F}}}{\overset{\eta_{\mathsf{T}\mathscr{F}}}{\rightrightarrows}} \mathsf{T}^2\mathscr{F}$$

where η is the unit of the adjunction $p^* \dashv p_*$. As a consequence of 3.8 $\mathsf{T}\mathscr{F}$, $\mathsf{T}^2\mathscr{F} \in \mathsf{Crt}(\mathbf{X})$, it follows that $\mathscr{F} \in \mathsf{Crt}(\mathbf{X})$. $\square$

Theorem 3.11. *Let $\mathbf{X}$ be a geometric stack. The subcategories $\mathsf{Crt}(\mathbf{X})$ and $\mathsf{Qco}(\mathbf{X})$ of $\mathscr{O}_{\mathbf{X}}$-Mod agree.*

Proof. Let us prove first that $\mathsf{Crt}(\mathbf{X}) \subset \mathsf{Qco}(\mathbf{X})$.

Consider a presentation $p\colon X \to \mathbf{X}$ as before and $\mathscr{F} \in \mathsf{Crt}(\mathbf{X})$. Let $X = \operatorname{Spec}(A_0)$. First, notice that the Cartesian sheaf $p^*\mathscr{F}$ sits in an exact sequence

$$\mathscr{O}_X^{(I)} \longrightarrow \mathscr{O}_X^{(J)} \longrightarrow p^*\mathscr{F} \longrightarrow 0 \tag{3.11.1}$$

that comes from a presentation of $M := \Gamma(X, p^*\mathscr{F})$ as A_0-module applying the functor $(-)_{\mathsf{crt}}$ of Proposition 2.2. Let $(U,u) \in \mathsf{Aff}_{\mathsf{fppf}}/\mathbf{X}$, and put $V = U \times_{\mathbf{X}} X$, p_1 and p_2 being the projections and $\phi\colon up_1 \Rightarrow pp_2$ the canonical 2-morphism of the pullback. We have the covering $\{(p_1,\phi^{-1})\colon (V,pp_2) \to (U,u)\}$ in $\mathsf{Aff}_{\mathsf{fppf}}/\mathbf{X}$ with $p_2\colon V \to X$ a flat finite presentation map. Applying p_2^* to the resolution (3.11.1) we obtain the exact sequence of sheaves of $\mathscr{O}_V$-modules

$$\mathscr{O}_V^{(I)} \longrightarrow \mathscr{O}_V^{(J)} \longrightarrow p_2^*p^*\mathscr{F} \longrightarrow 0.$$

The result follows from the fact that the category $\mathscr{O}_V$-Mod is equivalent to the category of sheaves of $\mathscr{O}|_{(V,pp_2)}$-modules over the site $(\mathsf{Aff}_{\mathsf{fppf}}/\mathbf{X})/(V,pp_2)$ by the comparison result [SGA 4_1, III, Théorème 4.1].

Let us prove now that $\mathsf{Qco}(\mathbf{X}) \subset \mathsf{Crt}(\mathbf{X})$.

Since $\mathscr{F} \in \mathsf{Qco}(\mathbf{X})$, there exists a covering $\{(f_i, \alpha_i)\colon (U_i, u_i) \to (X, p)\}_{i \in I}$ such that for the restriction of $\mathscr{F}$ to each $(\mathsf{Aff}_{\mathsf{fppf}}/\mathbf{X})/(U_i, u_i)$ there is a presentation

$$\mathscr{O}|^{(I)}_{(U_i,u_i)} \longrightarrow \mathscr{O}|^{(J)}_{(U_i,u_i)} \longrightarrow \mathscr{F}|_{(U_i,u_i)} \longrightarrow 0$$

i.e. a presentation of $\mathscr{O}_{U_i}$-modules

$$\mathscr{O}^{(I)}_{U_i} \longrightarrow \mathscr{O}^{(J)}_{U_i} \longrightarrow u_i^*\mathscr{F} \longrightarrow 0$$

$u_i^*\mathscr{F}$ being the sheaf of $\mathscr{O}_{U_i}$-modules given by $u_i^*\mathscr{F}(W,w) = \mathscr{F}(W, u_i w)$ for $(W,w) \in \mathsf{Aff}_{\mathsf{fppf}}/U_i$. The 2-morphism $\alpha_i\colon u_i \Rightarrow p f_i$ induces an isomorphism between the sheaves $u_i^*\mathscr{F}$ and $f_i^* p^*\mathscr{F}$. Since $\mathscr{O}_{U_i} \in \mathsf{Crt}(U_i)$, then $u_i^*\mathscr{F} \in \mathsf{Crt}(U_i)$ and thus $f_i^* p^*\mathscr{F} \in \mathsf{Crt}(U_i)$.

Because $p^*\mathscr{F}$ is a sheaf and $\{f_i\colon U_i \to X\}_{i\in I}$ is a covering in $\mathsf{Aff}_{\mathsf{fppf}}/X$ we obtain the equalizer of $\mathscr{O}_X$-modules

$$p^*\mathscr{F} \longrightarrow \textstyle\prod_{i\in I} f_{i*} f_i^* p^*\mathscr{F} \rightrightarrows \textstyle\prod_{i,j\in I} f_{ij*} f_{ij}^* p^*\mathscr{F},$$

where $f_{ij}\colon U_i \times_X U_j \to X$ is the composition of the first projection with f_i, or, what amounts to the same, the composition of the second projection with f_j. Since $f_{i*} f_i^* p^*\mathscr{F}, f_{ij*} f_{ij}^* p^*\mathscr{F} \in \mathsf{Crt}(X)$, then $p^*\mathscr{F} \in \mathsf{Crt}(X)$. The result follows from Lemma 3.10. □

Remark. The consequence of the previous theorem is that not only all quasi-coherent sheaves are Cartesian (presheaves) but that this fact characterizes them on a geometric stack. From now on, we will use freely this identification when dealing with quasi-coherent sheaves and use the most convenient characterization for the issue at hand.

4. Descent of quasi-coherent sheaves on geometric stacks

We proceed in our goal of describing quasi-coherent sheaves through algebraic data on an affine groupoid scheme whose stackification is the initial geometric stack. As a first step, we will develop a descent result for quasi-coherent sheaves with respect to presentations of stacks.

4.1. The category of descent data. Let $\mathbf{X}$ be a geometric stack and $p\colon X \to \mathbf{X}$ a presentation with X an affine scheme and $\mathscr{G}$ an $\mathscr{O}_X$-Module. As before, for an object $(U,u) \in \mathsf{Aff}_{\mathsf{fppf}}/\mathbf{X}$, we will abbreviate $\mathscr{G}(U,u)$ by $\mathscr{G}(U)$ if no confusion arises. Analogously, for a morphism (h,α) we will shorten $\mathscr{G}(h,\alpha)$ by $\mathscr{G}(h)$.

Consider the basic pull-back diagram

$$\begin{tikzcd} X \times_{\mathbf{X}} X \arrow[r, "p_2"] \arrow[d, "p_1"'] & X \arrow[d, "p"] \\ X \arrow[r, "p"'] \arrow[ur, Rightarrow, "\phi"] & \mathbf{X} \end{tikzcd} \tag{4.1.1}$$

A descent datum on $\mathscr{G}$ is an isomorphism $t\colon p_2^*\mathscr{G} \to p_1^*\mathscr{G}$ of $\mathscr{O}_{X\times_{\mathsf{X}}X}$-Modules verifying the cocycle condition

$$p_{12}^*t \circ p_{23}^*t = p_{13}^*t$$

on $\mathsf{Aff}_{\mathsf{fppf}}/X \times_{\mathsf{X}} X \times_{\mathsf{X}} X$, where as usual $p_{ij}\colon X \times_{\mathsf{X}} X \times_{\mathsf{X}} X \to X \times_{\mathsf{X}} X$ ($i < j \in \{1,2,3\}$) denotes the several projections obtained omitting the factor not depicted. We define the category $\mathsf{Desc}(X/\mathsf{X})$ by taking as its objects the pairs $(\mathscr{G}, t)$ with $\mathscr{G}$ an $\mathscr{O}_X$-Module and $t\colon p_2^*\mathscr{G} \to p_1^*\mathscr{G}$ a descent datum on $\mathscr{G}$; and its morphisms $h\colon (\mathscr{G},t) \to (\mathscr{G}',t')$ with $h\colon \mathscr{G} \to \mathscr{G}'$ a morphism of $\mathscr{O}_X$-Modules such that the diagram

$$\begin{tikzcd}
p_2^*\mathscr{G} \arrow[r, "t"] \arrow[d, "p_2^*h"'] & p_1^*\mathscr{G} \arrow[d, "p_1^*h"] \\
p_2^*\mathscr{G}' \arrow[r, "t'"] & p_1^*\mathscr{G}'
\end{tikzcd}$$

commutes.

4.2. Descent datum associated to a sheaf of modules. The 2-morphism $\phi\colon pp_1 \Rightarrow pp_2$ induces an isomorphism $\phi^*\colon p_2^*p^* \to p_1^*p^*$ given by

$$(\phi^*_{\mathscr{F}})(U,u) = \mathscr{F}(\mathrm{id}_U, \phi u), \text{ for } \mathscr{F} \in \mathscr{O}_{\mathsf{X}}\text{-}\mathsf{Mod} \text{ and } (U,u) \in \mathsf{Aff}_{\mathsf{fppf}}/X \times_{\mathsf{X}} X.$$

Notice that for any $\mathscr{O}_{\mathsf{X}}$-Module $\mathscr{F}$, it holds that $(p^*\mathscr{F}, \phi^*_{\mathscr{F}}) \in \mathsf{Desc}(X/\mathsf{X})$. Indeed, consider the pullback

$$\begin{tikzcd}
X \times_{\mathsf{X}} X \times_{\mathsf{X}} X \arrow[r, "p_3"] \arrow[d, "p_{12}"'] & X \arrow[d, "p"] \\
X \times_{\mathsf{X}} X \arrow[r, "pp_2"] \arrow[ur, Rightarrow, "\phi'"] & \mathsf{X}
\end{tikzcd} \tag{4.2.1}$$

With the previous notations, we have that $\phi p_{13} = \phi' \circ \phi p_{12}$ and $\phi p_{23} = \phi'$ and thus, $\phi p_{23} \circ \phi p_{12} = \phi p_{13}$. Now for $(U,u) \in \mathsf{Aff}_{\mathsf{fppf}}/X \times_{\mathsf{X}} X \times_{\mathsf{X}} X$ we have

$$\begin{aligned}
(p_{12}^*\phi^*_{\mathscr{F}} \circ p_{23}^*\phi^*_{\mathscr{F}})(U,u) &= \mathscr{F}(\mathrm{id}_U, \phi p_{12}u)\mathscr{F}(\mathrm{id}_U, \phi p_{23}u)\\
&= \mathscr{F}(\mathrm{id}_U, \phi p_{13}u)\\
&= (p_{13}^*\phi^*_{\mathscr{F}})(U,u).
\end{aligned}$$

This construction defines a functor

$$\mathsf{D}\colon \mathscr{O}_{\mathsf{X}}\text{-}\mathsf{Mod} \longrightarrow \mathsf{Desc}(X/\mathsf{X})$$

by $\mathsf{D}(\mathscr{F}) := (p^*\mathscr{F}, \phi^*_{\mathscr{F}})$.

4.3. From descent data to sheaves of modules. We define a functor

$$\mathsf{G}\colon \mathsf{Desc}(X/\mathsf{X}) \longrightarrow \mathscr{O}_{\mathsf{X}}\text{-}\mathsf{Mod}$$

assigning to $(\mathscr{G}, t) \in \mathsf{Desc}(X/\mathsf{X})$ the equalizer of the pair of morphisms

$$p_*\mathscr{G} \underset{b_{(\mathscr{G},t)}}{\overset{a_{\mathscr{G}}}{\rightrightarrows}} p_*p_{2*}p_1^*\mathscr{G}$$

where

$$a_{\mathscr{G}} := ((\phi_*)_{p_1^*\mathscr{G}}) \circ (p_*\eta_{1\mathscr{G}})$$
$$b_{(\mathscr{G},t)} := (p_*p_{2*}t) \circ (p_*\eta_{2\mathscr{G}})$$

with $\phi_*\colon p_*p_{1*} \xrightarrow{\sim} p_*p_{2*}$ denoting the functorial isomorphism and η_i the unit of the adjunction $p_i^* \dashv p_{i*}$ for $i \in \{1,2\}$. We denote this kernel by $\mathsf{G}(\mathscr{G},t)$. This construction is clearly functorial.

Proposition 4.4. *The previously defined functors are adjoint,* $\mathsf{D} \dashv \mathsf{G}$.

Proof. Let $i_{\mathscr{G}}\colon \mathsf{G}(\mathscr{G},t) \to p_*\mathscr{G}$ be the canonical morphism. For each morphism of $\mathscr{O}_{\mathbf{X}}$-modules $g\colon \mathscr{F} \to \mathsf{G}(\mathscr{G},t)$, the adjoint to $i_{\mathscr{G}}g\colon \mathscr{F} \to p_*\mathscr{G}$ through $p^* \dashv p_*$ provides the claimed morphism $(p^*\mathscr{F}, \phi^*_{\mathscr{F}}) \to (\mathscr{G},t)$ of descent data.

Reciprocally, if $f\colon \mathsf{D}(\mathscr{F}) \to (\mathscr{G},t)$ is a morphism in $\mathsf{Desc}(X/\mathbf{X})$, the morphism $\mathscr{F} \to p_*\mathscr{G}$ given by the adjunction $p^* \dashv p_*$ factors through $\mathsf{G}(\mathscr{G},t)$ giving a morphism $\mathscr{F} \to \mathsf{G}(\mathscr{G},t)$, as desired. □

To establish the next main result, Theorem 4.6, we will need a few technical preparations. Let $(V,v) \in \mathsf{Aff}_{\mathsf{fppf}}/X$ and consider the following pull-backs

$$\begin{tikzcd} V \times_{p_2} (X \times_{\mathbf{X}} X) \arrow[r, "v_2"] \arrow[d, "v_1"] & X \times_{\mathbf{X}} X \arrow[d, "p_2"] \\ V \arrow[r, "v"] & X \end{tikzcd} \qquad \begin{tikzcd} V \times_{\mathbf{X}} X \arrow[r, "w_2"] \arrow[d, "w_1"] & X \arrow[d, "p"] \\ V \arrow[r, "pv"] \arrow[ur, Rightarrow, "\alpha"] & \mathbf{X} \end{tikzcd}$$

Denote $W = V \times_{p_2} (X \times_{\mathbf{X}} X)$. Set $h\colon V \times_{\mathbf{X}} X \to X \times_{\mathbf{X}} X$ the morphism such that $p_1 h = w_2$, $p_2 h = v w_1$ and $\phi h = \alpha^{-1}$. The morphism $h'\colon V \times_{\mathbf{X}} X \to W$ verifying $v_1 h' = w_1$ and $v_2 h' = h$ is an isomorphism. Let us define

$$\theta\colon p_{2*}p_1^* \longrightarrow p^*p_*$$

the isomorphism of functors given by $(\theta_{\mathscr{G}})(V,v) = \mathscr{G}(h')$ for each $\mathscr{G} \in \mathscr{O}_X\text{-}\mathsf{Mod}$. Set

$$c_{(\mathscr{G},t)} := \theta_{\mathscr{G}} \circ (p_{2*}t) \circ \eta_{2\mathscr{G}}.$$

Lemma 4.5. *With the previous notations, the following identity holds*

$$p^*a_{\mathscr{G}} \circ c_{(\mathscr{G},t)} = p^*b_{(\mathscr{G},t)} \circ c_{(\mathscr{G},t)}.$$

Proof. Consider the pull-backs

$$\begin{tikzcd} W_1 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & X \times_{\mathbf{X}} X \arrow[d, "p_1"] \\ V \times_{\mathbf{X}} X \arrow[r, "w_2"] & X \end{tikzcd} \qquad \begin{tikzcd} W_2 \arrow[r, "\pi'_2"] \arrow[d, "\pi'_1"'] & X \times_{\mathbf{X}} X \arrow[d, "p_2"] \\ V \times_{\mathbf{X}} X \arrow[r, "w_2"] & X \end{tikzcd}$$

Let $l\colon W_2 \to V \times_{\mathbf{X}} X$ be the morphism such that $w_1 l = w_1 \pi'_1$, $w_2 l = p_1 \pi'_2$ and $\alpha l = \phi^{-1}\pi'_2 \circ \alpha\pi'_1$. Put $l'\colon W_2 \to W_1$ for the morphism verifying that $\pi_1 l' = l$

and $\pi_2 l' = \pi'_2$. Notice that l' is an isomorphism and $p^*((\phi_*)_{p_1^*\mathscr{G}})(V,v) = \mathscr{G}(l')$. We have

$$\begin{aligned}(p^*((\phi_*)_{p_1^*\mathscr{G}} \circ p_*\eta_{1\mathscr{G}}) \circ c_{(\mathscr{G},t)}))(V,v) &= \mathscr{G}(l')\mathscr{G}(\pi_1)\mathscr{G}(h')t(W,v_2)\mathscr{G}(v_1)\\ &= \mathscr{G}(h'l)t(W,v_2)\mathscr{G}(v_1)\\ &= t(W_2,hl)\mathscr{G}(h'l)\mathscr{G}(v_1)\\ &= t(W_2,hl)\mathscr{G}(w_1 l).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}(p^*(p_*p_{2*}t \circ p_*\eta_{2\mathscr{G}}) \circ c_{(\mathscr{G},t)})(V,v) &= t(W_2,\pi'_2)\mathscr{G}(\pi'_1)\mathscr{G}(h')t(W,v_2)\mathscr{G}(v_1)\\ &= t(W_2,\pi'_2)t(W_2,h\pi'_1)\mathscr{G}(h'\pi'_1)\mathscr{G}(v_1)\\ &= t(W_2,\pi'_2)t(W_2,h\pi'_1)\mathscr{G}(w_1 l).\end{aligned}$$

Thus, it is sufficient to prove that $t(W_2,\pi'_2) \circ t(W_2,h\pi'_1) = t(W_2,hl)$. For that, consider the 2-pull-back

$$\begin{tikzcd}(X \times_{\mathsf{X}} X)_{pp_2} \times_{\mathsf{X}} V \arrow[r, "c_2"] \arrow[d, "c_1"'] & V \arrow[d, "pv"] \\ X \times_{\mathsf{X}} X \arrow[r, "pp_2"'] \arrow[ur, Rightarrow, "\gamma"] & \mathsf{X}\end{tikzcd}$$

Let $k\colon W_2 \to (X \times_{\mathsf{X}} X)_{pp_2} \times_{\mathsf{X}} V$ be the isomorphism given by $c_1 k = \pi'_2$, $c_2 k = w_1\pi'_1$, and $\gamma k = \alpha^{-1}\pi'_1$. Consider the morphism

$$\mathrm{id} \times_{\mathsf{X}} v : (X \times_{\mathsf{X}} X)_{pp_2} \times_{\mathsf{X}} V \longrightarrow X \times_{\mathsf{X}} X \times_{\mathsf{X}} X$$

satisfying $p_{12}(\mathrm{id} \times_{\mathsf{X}} v) = c_1$, $p_3(\mathrm{id} \times_{\mathsf{X}} v) = vc_2$, and $\phi'(\mathrm{id} \times_{\mathsf{X}} v) = \gamma$. The morphism $g = (\mathrm{id} \times_{\mathsf{X}} v)k$ is a finitely presented flat map. Applying the cocycle condition of t to (W_2, g), we obtain that

$$t(W_2, p_{12}g) \circ t(W_2, p_{23}g) = t(W_2, p_{13}g)$$

and the result follows because $p_{12}g = \pi'_2$, $p_{23}g = h\pi'_1$ and $p_{13}g = hl$. □

Theorem 4.6. *The functor* $\mathsf{D}\colon \mathscr{O}_{\mathsf{X}}\text{-}\mathsf{Mod} \to \mathsf{Desc}(X/\mathsf{X})$ *(and, therefore its adjoint* G*) is an equivalence of categories.*

Proof. We will see first that $\bar{\eta}$, the unit of the adjunction $\mathsf{D} \dashv \mathsf{G}$, is an isomorphism. Let $\mathscr{F} \in \mathscr{O}_{\mathsf{X}}\text{-}\mathsf{Mod}$ and consider the following commutative diagram

$$\begin{tikzcd}\mathscr{F} \arrow[r, "\eta_{\mathscr{F}}"] \arrow[d, "\bar{\eta}_{\mathscr{F}}"'] & p_*p^*\mathscr{F} \arrow[r, shift left, "\rho_1"] \arrow[r, shift right, "\rho_2"'] \arrow[d, equal] & p_*p^*p_*p^*\mathscr{F} \arrow[d, "\text{via } \theta^{-1}"] \\ \mathsf{G}(p^*\mathscr{F}, \phi^*_{\mathscr{F}}) \arrow[r, "i_{p^*\mathscr{F}}"] & p_*p^*\mathscr{F} \arrow[r, shift left, "a_{p^*\mathscr{F}}"] \arrow[r, shift right, "b_{(p^*\mathscr{F},\phi^*_{\mathscr{F}})}"'] & p_*p_{2*}p_1^*p^*\mathscr{F}\end{tikzcd}$$

where $\eta_{\mathscr{F}}$ is the unit of the adjunction $p^* \dashv p_*$ applied to $\mathscr{F}$. The first row is an equalizer because $\mathscr{F}$ is a sheaf and the claim follows.

Let us prove now that $\bar{\varepsilon}$, the counit of the adjunction $\mathsf{D} \dashv \mathsf{G}$, is an isomorphism. Let $(\mathscr{G}, t) \in \mathsf{Desc}(X/\mathbf{X})$. We have that $\bar{\varepsilon}_{(\mathscr{G},t)} = \varepsilon_{\mathscr{G}} \circ p^* i_{\mathscr{G}}$, where $\varepsilon_{\mathscr{G}}$ denotes the counit of the adjunction $p^* \dashv p_*$.

$$\begin{tikzcd}
p^*\mathsf{G}(\mathscr{G},t) \arrow[r, "p^* i_{\mathscr{G}}"] \arrow[d, "\bar{\varepsilon}_{(\mathscr{G},t)}"'] & p^* p_* \mathscr{G} \arrow[r, shift left, "p^* a_{\mathscr{G}}"] \arrow[r, shift right, "p^* b_{(\mathscr{G},t)}"'] & p^* p_* p_{2*} p_1^* \mathscr{G} \\
\mathscr{G} \arrow[ur, "c_{(\mathscr{G},t)}"'] & &
\end{tikzcd}$$

Since p^* preserve finite limits, the horizontal row is exact and so by Lemma 4.5 there exists a morphism $r\colon \mathscr{G} \to p^*\mathsf{G}(\mathscr{G},t)$ with $p^* i_{\mathscr{G}} \circ r = c_{(\mathscr{G},t)}$. It holds that $c_{(\mathscr{G},t)}\, \bar{\varepsilon}_{(\mathscr{G},t)} = p^* i_{\mathscr{G}}$. Indeed,

$$\begin{aligned}
c_{(\mathscr{G},t)} \circ \bar{\varepsilon}_{(\mathscr{G},\iota)} &= \theta_{\mathscr{G}} \circ p_{2*} t \circ \eta_{2\mathscr{G}} \circ \varepsilon_{\mathscr{G}} \circ p^* i_{\mathscr{G}} \\
&= \varepsilon_{p^* p_* \mathscr{G}} \circ p^* p_* (\theta_{\mathscr{G}} \circ p_{2*} t \circ \eta_{2\mathscr{G}}) \circ p^* i_{\mathscr{G}} \\
&\cong \varepsilon_{p^* p_* \mathscr{G}} \circ p^* p_* \theta_{\mathscr{G}} \circ p^*((\phi_*)_{p_1^* \mathscr{G}}) \circ p^* p_* \eta_{1\mathscr{G}} \circ p^* i_{\mathscr{G}}.
\end{aligned}$$

and after an straightforward calculation we see that

$$\varepsilon_{p^* p_* \mathscr{G}} \circ p^* p_* \theta_{\mathscr{G}} \circ p^*((\phi_*)_{p_1^* \mathscr{G}}) \circ p^* p_* \eta_{1\mathscr{G}} = \mathrm{id}_{p^* p_* \mathscr{G}}.$$

We see now that r is the inverse of $\bar{\varepsilon}_{(\mathscr{G},t)}$. Indeed,

$$p^* i_{\mathscr{G}} \circ r \circ \bar{\varepsilon}_{(\mathscr{G},t)} = c_{(\mathscr{G},t)} \circ \bar{\varepsilon}_{(\mathscr{G},t)} = p^* i_{\mathscr{G}}$$

and $p^* i_{\mathscr{G}}$ is an monomorphism of sheaves, therefore $r \circ \bar{\varepsilon}_{(\mathscr{G},t)} = \mathrm{id}_{\mathscr{G}}$. Since $\mathscr{G}$ is a sheaf, $\eta_{2\mathscr{G}}$ is injective and thus $c_{(\mathscr{G},t)}$ is an monomorphism of sheaves. From

$$c_{(\mathscr{G},t)} \circ \bar{\varepsilon}_{(\mathscr{G},t)} \circ r = p^* i_{\mathscr{G}} \circ r = c_{(\mathscr{G},t)}$$

it follows that $\bar{\varepsilon}_{(\mathscr{G},t)} \circ r = \mathrm{id}_{p^*\mathsf{G}(\mathscr{G},t)}$. □

We denote by $\mathsf{Desc}_{\mathsf{qc}}(X/\mathbf{X})$ the full subcategory of the category $\mathsf{Desc}(X/\mathbf{X})$ whose objects are the pairs $(\mathscr{G}, t)$ with $\mathscr{G} \in \mathsf{Qco}(X)$.

Corollary 4.7. *The categories* $\mathsf{Qco}(\mathbf{X})$ *and* $\mathsf{Desc}_{\mathsf{qc}}(X/\mathbf{X})$ *are equivalent.*

Proof. The result follows because for $\mathscr{F} \in \mathsf{Qco}(\mathbf{X})$ we have $\mathsf{D}(\mathscr{F}) \in \mathsf{Desc}_{\mathsf{qc}}(X/\mathbf{X})$ and, conversely, for $(\mathscr{G}, t) \in \mathsf{Desc}_{\mathsf{qc}}(X/\mathbf{X})$ it holds that $\mathsf{G}(\mathscr{G}, t) \in \mathsf{Qco}(\mathbf{X})$ as follows from Propositions 3.7, 3.8 and 4.6, having in mind Theorem 3.11. □

Remark. Our method of proof is not too distant from Olsson's [O, §4]. Notice, however, that he is working with Cartesian sheaves on a simplicial site while our result follows from the more general Theorem 4.6 that applies not only to quasi-coherent Modules.

5. Representing quasi-coherent sheaves by comodules

5.1. Hopf algebroids as affine models of geometric stacks. Let $\mathbf{X}$ be a geometric stack and $p\colon X \to \mathbf{X}$ a presentation with X affine. The fibered product $X \times_{\mathbf{X}} X$ is an affine scheme because $\mathbf{X}$ is geometric. As we recalled in §3, the pair $(X, X \times_{\mathbf{X}} X)$ is a scheme in groupoids whose associated stack is $\mathbf{X}$.

Let A_0 and A_1 be the rings defined by $X = \operatorname{Spec}(A_0)$ and $X_1 := X \times_{\mathbf{X}} X = \operatorname{Spec}(A_1)$. Denote the pair $A_\bullet := (A_0, A_1)$. The dual structure of a scheme in groupoids is called a *Hopf algebroid*. Let us spell out the corresponding structure of $A_\bullet$, there are:

- *two* homomorphisms
$$\eta_L, \eta_R \colon A_0 \rightrightarrows A_1,$$
corresponding respectively to the projections $p_1, p_2 \colon X \times_{\mathbf{X}} X \rightrightarrows X$;
- the *counit* homomorphism
$$\epsilon\colon A_1 \longrightarrow A_0,$$
corresponding to the diagonal $\delta\colon X \to X \times_{\mathbf{X}} X$;
- the *conjugation* homomorphism
$$\kappa\colon A_1 \longrightarrow A_1,$$
corresponding to the map interchanging the factors in $X \times_{\mathbf{X}} X$;
- and the *comultiplication*
$$\nabla\colon A_1 \longrightarrow A_1 {}_{\eta_R}\!\otimes_{\eta_L} A_1,$$
corresponding, via the isomorphism
$$X \times_{\mathbf{X}} X \times_{\mathbf{X}} X \xrightarrow{\sim} (X \times_{\mathbf{X}} X)\, {}_{p_1}\!\times_{p_2} (X \times_{\mathbf{X}} X)$$
induced by the projections p_{23} and p_{12}, to the projection
$$p_{13}\colon X \times_{\mathbf{X}} X \times_{\mathbf{X}} X \to X \times_{\mathbf{X}} X,$$
expressing the composition of arrows in the scheme in groupoids $(X, X \times_{\mathbf{X}} X)$.

These data is subject to the following identities:

(*i*) $\epsilon\eta_L = \mathrm{id}_{A_0} = \epsilon\eta_R$ (source and target of identity).
(*ii*) If $j_1, j_2 \colon A_1 \to A_1 {}_{\eta_R}\!\otimes_{\eta_L} A_1$ are the maps given by $j_1(b) = b \otimes 1$, $j_2(b) = 1 \otimes b$, then $\nabla\eta_L = j_1\eta_L$ and $\nabla\eta_R = j_2\eta_R$ (source and target of a composition).
(*iii*) $\kappa\eta_L = \eta_R$ and $\kappa\eta_R = \eta_L$ (source and target of inverse).
(*iv*) $(\mathrm{id}_{A_1} \otimes \epsilon)\nabla = \mathrm{id}_{A_1} = (\epsilon \otimes \mathrm{id}_{A_1})\nabla$ (identity).
(*v*) $(\mathrm{id}_{A_1} \otimes \nabla)\nabla = (\nabla \otimes \mathrm{id}_{A_1})\nabla$ (associativity of composition).
(*vi*) If μ is the multiplication on A_1, then $\mu(\kappa \otimes \mathrm{id}_{A_1})\nabla = \eta_R\epsilon$ and $\mu(\mathrm{id}_{A_1} \otimes \kappa)\nabla = \eta_L\epsilon$ (inverse).
(*vii*) $\kappa\kappa = \mathrm{id}_{A_1}$ (inverting twice).

Compare with [R, A1.1.1.] and [LMB, (2.4.3)].

The geometric stack defined by a scheme in groupoids $(\mathrm{Spec}(A_0), \mathrm{Spec}(A_1))$ is denoted

$$\mathsf{Stck}(A_\bullet) := [(\mathrm{Spec}(A_0), \mathrm{Spec}(A_1))]$$

(with the right hand side notation as in [LMB, (3.4.3)]). So, with the notation at the beginning of this section, $\mathbf{X} = \mathsf{Stck}(A_\bullet)$.

From now on, every geometric stack we consider will be the stack associated to a Hopf algebroid. From a geometric point of view, the data of a Hopf algebroid is more rigid than the underlying algebraic stack alone. It is equivalent to the datum of an algebraic stack *together with a smooth presentation* by an affine scheme. In any case, the algebroid determines the stack and we will use this fact in what follows. Note that κ is an isomorphism (being inverse to itself) and that η_L and η_R are *smooth* morphisms.

5.2. Comodules over a Hopf algebroid. Let $A_\bullet := (A_0, A_1)$ be a Hopf algebroid. A (left) $A_\bullet$-comodule (M, ψ_M) is an A_0-module M together with an A_0-linear map $\psi_M \colon M \to A_1\,{}_{\eta_R}\!\otimes_{A_0} M$, the target being a A_0-module via η_L, and verifying the following identities:

(*i*) $(\nabla \otimes \mathrm{id}_M)\psi_M = (\mathrm{id}_{A_1} \otimes \psi_M)\psi_M$ (coassociativity of ψ).
(*ii*) $(\epsilon \otimes \mathrm{id}_M)\psi_M = \mathrm{id}_M$ (ψ_M is counitary).

The map ψ_M is called the comodule structure map of M. If M and M' are (left) $A_\bullet$-comodules, a map of $A_\bullet$-comodules is a A_0-linear map $\lambda \colon M \to M'$ such that the following square is commutative

$$\begin{tikzcd} M \arrow[r, "\psi_M"] \arrow[d, "\lambda"'] & A_1\,{}_{\eta_R}\!\otimes_{A_0} M \arrow[d, "\mathrm{id}\otimes\lambda"] \\ M' \arrow[r, "\psi_{M'}"] & A_1\,{}_{\eta_R}\!\otimes_{A_0} M' \end{tikzcd}$$

We denote by $A_\bullet$-coMod the category of (left) $A_\bullet$-comodules. All comodules considered in this paper will be on the left so we will speak simply of comodules.

If M is an A_0-module one can always define a structure of $A_\bullet$-comodule on $A_1\,{}_{\eta_R}\!\otimes_{A_0} M$ by defining $\psi_{A_1\,{}_{\eta_R}\!\otimes_{A_0} M} := \nabla \otimes \mathrm{id}$. This structure is called the *extended comodule* structure on $A_1\,{}_{\eta_R}\!\otimes_{A_0} M$.

Theorem 5.3. *If A_1 is flat as an A_0-module, then the category $A_\bullet$-coMod is a Grothendieck category.*

Proof. In [R, A1.1.3.] it is proved that $A_\bullet$-coMod is an abelian category if A_1 is a flat A_0-module. Therefore what is left to prove is that this category has exact direct limits and a set of generators.

Since tensor product commutes with colimits, the A_0-module colimit of a diagram in $A_\bullet$-coMod is an $A_\bullet$-comodule. Direct limits of $A_\bullet$-comodules are exact, because direct limits are exact in the category A_0-Mod. Since A_1 is flat

as an A_0-module, the kernel as an A_0-module of a $A_\bullet$-comodule morphism inherits the structure of $A_\bullet$-comodule.

Finally we show that the set S of $A_\bullet$-subcomodules of A_1^n for $n \in \mathbb{N}$ (viewed as $A_\bullet$-comodules via ∇) is a set of generators of $A_\bullet$-coMod. Indeed, let M be a left $A_\bullet$-comodule, $M \neq 0$. The structure morphism $\psi = \psi_M : M \to A_1\, {}_{\eta_R}\!\otimes_{A_0} M$ is injective as a morphism of $A_\bullet$-comodules, considering the structure of extended comodule on $A_1\, {}_{\eta_R}\!\otimes_{A_0} M$. Denote by $q : A_1\, {}_{\eta_R}\!\otimes_{A_0} M \to N$ the cokernel of ψ and let $p : A_0^{(I)} \to M$ be a free presentation of M as an A_0-module. Since the morphism $\mathrm{id} \otimes p : A_1\, {}_{\eta_R}\!\otimes_{A_0} A_0^{(I)} \to A_1\, {}_{\eta_R}\!\otimes_{A_0} M$ is a morphism of comodules and $A_1^{(I)} \cong A_1\, {}_{\eta_R}\!\otimes_{A_0} A_0^{(I)}$ as comodules, we have a surjective morphism of comodules $\bar{p} : A_1^{(I)} \to A_1\, {}_{\eta_R}\!\otimes_{A_0} M$. Given $x \in M$, $x \neq 0$, there are $n \in \mathbb{N}$, $y \in A_1^n$ such that $\bar{p}\, j(y) = \psi(x)$ where $j : A_1^n \to A_1^{(I)}$ is the corresponding canonical morphism. If K is the kernel of $q \bar{p} j : A_1^n \to N$, $K \in \mathsf{S}$ and $y \in K$, then $(\bar{p} j)|_K$ factors through a non-zero map, *i.e.* $\mathrm{Hom}_{A_\bullet\text{-}\mathsf{coMod}}(K, M) \neq 0$. $\square$

Remark. We remind the reader that if $A_\bullet$ is a Hopf algebroid arising from a geometric stack, A_1 is smooth over A_0 and, therefore, flat.

5.4. The category of descent data in $A_\bullet$. Let $A_\bullet$ be a Hopf algebroid as before, and M an A_0-module. Write $A_2 = A_1\, {}_{\eta_R}\!\otimes_{\eta_L} A_1$. A descent datum on M is an isomorphism $\tau : A_1\, {}_{\eta_L}\!\otimes_{A_0} M \to A_1\, {}_{\eta_R}\!\otimes_{A_0} M$ of A_1-modules veryfying that the diagram

$$
\begin{tikzcd}
A_2\, {}_{j_1\eta_L}\!\otimes_{A_0} M \arrow[r, equal] \arrow[d, equal] & A_2\, {}_{j_1}\!\otimes_{A_1} A_1\, {}_{\eta_L}\!\otimes_{A_0} M \arrow[r, "\mathrm{id}\otimes\tau"] & A_2\, {}_{j_1}\!\otimes_{A_1} A_1\, {}_{\eta_R}\!\otimes_{A_0} M \arrow[dd, "\wr", "j_1\eta_R = j_2\eta_L"'] \\
A_2\, {}_{\nabla}\!\otimes_{A_1} A_1\, {}_{\eta_L}\!\otimes_{A_0} M \arrow[d, "\mathrm{id}\otimes\tau"'] & & \\
A_2\, {}_{\nabla}\!\otimes_{A_1} A_1\, {}_{\eta_R}\!\otimes_{A_0} M \arrow[d, equal] & & A_2\, {}_{j_2}\!\otimes_{A_1} A_1\, {}_{\eta_L}\!\otimes_{A_0} M \arrow[d, "\mathrm{id}\otimes\tau"] \\
A_2\, {}_{\nabla\eta_R}\!\otimes_{A_0} M \arrow[rr, equal] & & A_2\, {}_{j_2}\!\otimes_{A_1} A_1\, {}_{\eta_R}\!\otimes_{A_0} M
\end{tikzcd}
\tag{5.4.1}
$$

commutes.

We define the category $\mathrm{Desc}(A_\bullet)$ by taking as its objects the pairs (M, τ) with M an A_0-module and τ a descent datum on M, and as its morphisms $f : (M_1, \tau_1) \to (M_2, \tau_2)$ those homomorphism of A_0-modules $f : M_1 \to M_2$ such that the diagram

$$
\begin{tikzcd}
A_1\, {}_{\eta_L}\!\otimes_{A_0} M_1 \arrow[r, "\tau_1"] \arrow[d, "\mathrm{id}\otimes f"'] & A_1\, {}_{\eta_R}\!\otimes_{A_0} M_1 \arrow[d, "\mathrm{id}\otimes f"] \\
A_1\, {}_{\eta_L}\!\otimes_{A_0} M_2 \arrow[r, "\tau_2"] & A_1\, {}_{\eta_R}\!\otimes_{A_0} M_2
\end{tikzcd}
$$

commutes.

For future reference, note that commutativity of diagram (5.4.1) is equivalent to the following identity in $A_{1\,\eta_R}\otimes_{\eta_L} A_{1\,\eta_R}\otimes_{A_0} M$

$$\sum_{i=1}^{n} ac_i \otimes b\tau(1\otimes x_i) = \sum_{i=1}^{n} (a\otimes b)\nabla(c_i)\otimes x_i \tag{5.4.2}$$

where $a,b\in A_1$ and $x\in M$, with $\tau(1\otimes x)=\sum_{i=1}^{n} c_i\otimes x_i$.

Proposition 5.5. *The categories* $\mathsf{Desc}(A_\bullet)$ *and* $A_\bullet$-coMod *are isomorphic.*

Proof. Let $(M,\tau)\in\mathsf{Desc}(A_\bullet)$. To give a structure of $A_\bullet$-comodule on M one defines $\psi=\psi_M\colon M\to A_{1\,\eta_R}\otimes_{A_0} M$ as the following composition

$$M\xrightarrow{\eta_L\otimes\mathrm{id}} A_{1\,\eta_L}\otimes_{A_0} M\xrightarrow{\tau} A_{1\,\eta_R}\otimes_{A_0} M.$$

Let us see that ψ is counitary. Let $x\in M$, with notation as in 5.4.2 we have

$$(\epsilon\otimes\mathrm{id})\psi(x)=\sum_{i=1}^{n}\epsilon(c_i)x_i.$$

To see that $\sum_{i=1}^{n}\epsilon(c_i)x_i=x$, we apply the homomorphism $\epsilon\otimes\mathrm{id}\otimes\mathrm{id}$ to both sides of the identity (5.4.2) for $a=b=1$ to get

$$\sum_{i=1}^{n}\tau(\eta_L\epsilon(c_i)\otimes x_i)=\sum_{i=1}^{n}c_i\otimes x_i,$$

and since τ is an isomorphism we deduce that

$$\sum_{i=1}^{n}\eta_L\epsilon(c_i)\otimes x_i=1\otimes x.$$

Thus

$$\sum_{i=1}^{n}\epsilon(c_i)x_i=(\epsilon\otimes\mathrm{id})\sum_{i=1}^{n}\eta_L\epsilon(c_i)\otimes x_i=(\epsilon\otimes\mathrm{id})(1\otimes x)=x.$$

The coassociativity of ψ follows from the identity (5.4.2) for $a=b=1$. Now, if $f\colon(M_1,\tau_1)\to(M_2,\tau_2)$ is a morphim in $\mathsf{Desc}(A_\bullet)$, then

$$(\mathrm{id}\otimes f)\psi_1=(\mathrm{id}\otimes f)\tau_1(\eta_L\otimes\mathrm{id})=\tau_2(\mathrm{id}\otimes f)(\eta_L\otimes\mathrm{id})=\tau_2(\eta_L\otimes\mathrm{id})f=\psi_2 f$$

so $f\colon(M_1,\psi_1)\to(M_2,\psi_2)$ is a morphism of $A_\bullet$-comodules. It follows that this defines a functor $\mathsf{C}\colon\ \mathsf{Desc}(A_\bullet)\to A_\bullet$-$\mathsf{coMod}$.

Conversely, let (M,ψ) be an $A_\bullet$-comodule. We define

$$\tau\colon A_{1\,\eta_L}\otimes_{A_0}M\to A_{1\,\eta_R}\otimes_{A_0}M$$

as the composition

$$A_{1\,\eta_L}\otimes_{A_0}M\xrightarrow{\mathrm{id}\otimes\psi}A_{1\,\eta_L}\otimes_{\eta_L}A_{1\,\eta_R}\otimes_{A_0}M\xrightarrow{\mu\otimes\mathrm{id}}A_{1\,\eta_R}\otimes_{A_0}M.$$

It easy to prove that τ is an homomorphism of A_1-modules. We show that τ is an isomorphism by constructing its inverse. We define τ' as the following composition

$$A_{1\eta_R}\otimes_{A_0}M\xrightarrow{\mathrm{id}\otimes\psi}A_{1\eta_R}\otimes_{\eta_L}A_{1\eta_R}\otimes_{A_0}M\xrightarrow{\mathrm{id}\otimes\kappa\otimes\mathrm{id}}A_{1\eta_R}\otimes_{\eta_R}A_{1\eta_L}\otimes_{A_0}M\xrightarrow{\mu\otimes\mathrm{id}}A_{1\eta_L}\otimes_{A_0}M.$$

Let us see that τ and τ' are mutually inverse. Consider the commutative diagram in which all tensor products are taken over A_0 (with most subindices omitted)

$$\begin{tikzcd}
A_1{}_{\eta_R}\otimes M \arrow[r, "\mathrm{id}\otimes\psi"] \arrow[d, "\mathrm{id}\otimes\psi"] & A_1\otimes A_1\otimes M \arrow[r, "\mathrm{id}\otimes\kappa\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\mathrm{id}\otimes\psi"] & A_1\otimes A_1\otimes M \arrow[r, "\mu\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\mathrm{id}\otimes\psi"] & A_1{}_{\eta_L}\otimes M \arrow[d, "\mathrm{id}\otimes\psi"] \\
A_1\otimes A_1\otimes M \arrow[r, "\mathrm{id}\otimes\nabla\otimes\mathrm{id}"] & A_1\otimes A_1\otimes A_1\otimes M \arrow[r, "\mathrm{id}\otimes\kappa\otimes\mathrm{id}"] & A_1\otimes A_1\otimes A_1\otimes M \arrow[r, "\mu\otimes\mathrm{id}\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\mu\otimes\mathrm{id}"] & A_1\otimes A_1\otimes M \arrow[d, "\mu\otimes\mathrm{id}"] \\
& & A_1\otimes A_1\otimes M \arrow[r, "\mu\otimes\mathrm{id}"] & A_1{}_{\eta_R}\otimes M
\end{tikzcd}$$

then we have

$$\begin{aligned}
\tau\tau' &= (\mu\otimes\mathrm{id})(\mathrm{id}\otimes\mu\otimes\mathrm{id})(\mathrm{id}\otimes\kappa\otimes\mathrm{id})(\mathrm{id}\otimes\nabla\otimes\mathrm{id})(\mathrm{id}\otimes\psi)\\
&= (\mu\otimes\mathrm{id})(\mathrm{id}\otimes\mu(\kappa\otimes\mathrm{id})\nabla\otimes\mathrm{id})(\mathrm{id}\otimes\psi)\\
&= (\mu\otimes\mathrm{id})(\mathrm{id}\otimes\eta_R\epsilon\otimes\mathrm{id})(\mathrm{id}\otimes\psi) = \mathrm{id}.
\end{aligned}$$

A similar computation shows that $\tau'\tau = \mathrm{id}$.

Finally, it remains to prove that τ verifies (5.4.2). Since ψ is coassociative, we have for each $x \in M$

$$\sum_{i=1}^{n} c_i \otimes \psi(x_i) = \sum_{i=1}^{n} \nabla(c_i)\otimes x_i \tag{5.5.1}$$

where $\psi(x) = \sum_{i=1}^{n} c_i \otimes x_i$. But $\tau(1\otimes x) = \psi(x)$, for each $x \in M$, and the result follows multiplying by $a\otimes b \in A_2$ in both sides of (5.5.1).

Now, if $f\colon (M_1,\psi_1) \to (M_2,\psi_2)$ is a morphism of $A_\bullet$-comodules, and τ_1 and τ_2 are their corresponding descent data, then

$$(\mathrm{id}\otimes f)\tau_1 = (\mu\otimes\mathrm{id})(\mathrm{id}\otimes\mathrm{id}\otimes f)(\mathrm{id}\otimes\psi_1) = \tau_2(\mathrm{id}\otimes f)$$

and so $f\colon (M_1,\tau_1) \to (M_2,\tau_2)$ is a morphism in $\mathsf{Desc}(A_\bullet)$. It follows that this define a functor $\mathsf{D}'\colon A_\bullet\text{-}\mathsf{coMod} \to \mathsf{Desc}(A_\bullet)$.

Note that the functors C and D' are inverse to each other. In fact, let $(M,\tau) \in \mathsf{Desc}(A_\bullet)$ and $\psi\colon M \to A_1{}_{\eta_R}\otimes_{A_0} M$ be its associated $A_\bullet$-comodule structure. The descent data associated to the comodule (M,ψ) is the composition $(\mu\otimes\mathrm{id})(\mathrm{id}\otimes\psi) = \tau$. Conversely, let (M,ψ) be an $A_\bullet$-comodule. If τ is the corresponding descent data on M, the structure map of the $A_\bullet$-comodule associated to (M,τ) is the morphism $\tau(\eta_L\otimes 1) = \psi$. $\square$

5.6. Let $X = \mathrm{Spec}(A)$ be an affine scheme. Recall that by Theorem 3.11 the categories $\mathsf{Qco}(X)$ and $\mathsf{Crt}(X)$ are the same. Correspondingly, the functors $\widetilde{(-)}$ and $(-)_{\mathsf{crt}}$ are identified, too. We will stick to the former notation according to the usage of [EGA I, (1.3)].

Proposition 5.7. *The couple of functors*

$$\Gamma(X,-)\colon \mathsf{Qco}(X) \to A_0\text{-}\mathsf{Mod} \quad \textit{and} \quad \widetilde{(-)}\colon A_0\text{-}\mathsf{Mod} \to \mathsf{Qco}(X)$$

induces an equivalence of categories between $\mathsf{Desc}_{\mathsf{qc}}(X/\mathbf{X})$ *and* $\mathsf{Desc}(A_\bullet)$*, that we will keep denoting* $\Gamma(X,-)$ *and* $\widetilde{(-)}$.

Proof. If $(\mathscr{G},t) \in \mathsf{Desc}_{\mathsf{qc}}(X/\mathbf{X})$, we will prove that the morphism

$$\tau := (\mathscr{G}(p_2)^{\mathsf{a}})^{-1}\, t^{-1}(X_1, \mathrm{id}_{X_1})\, \mathscr{G}(p_1)^{\mathsf{a}}$$

is a descent datum on $\mathscr{G}(X)$.

Consider the Cartesian square

$$\begin{array}{ccc} X_2 & \xrightarrow{q_2} & X_1 \\ {\scriptstyle q_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ X_1 & \xrightarrow{p_1} & X \end{array}$$

where $X_2 = \operatorname{Spec}(A_1\, {}_{\eta_R}\!\otimes_{\eta_L} A_1)$, $q_1 = \operatorname{Spec}(j_2)$ and $q_2 = \operatorname{Spec}(j_1)$. The multiplication of the scheme in groupoids (X, X_1) is the morphism $m\colon X_2 \to X_1$ verifying $p_1 m = p_1 q_2$, $p_2 m = p_2 q_1$ and $\phi m = \phi q_1 \circ \phi q_2$, with notations as in (4.1.1). Let

$$w\colon X_2 \to X \times_{\mathbf{X}} X \times_{\mathbf{X}} X \tag{5.7.1}$$

be the isomorphism given by $p_{12} w = q_2$, $p_3 w = p_2 q_1$ and $\phi' w = \phi q_1$ with ϕ' as in 4.2.1. Its inverse w^{-1}, already defined on 5.1, is the unique morphism verifying $q_1 w^{-1} = p_{23}$ and $q_2 w^{-1} = p_{12}$. As a consequence

$$p_{12} w = q_2 \quad p_{23} w = q_1 \text{ and } p_{13} w = m.$$

Thus, the cocycle condition of t on $\mathscr{G}$ reads

$$q_2^* t \circ q_1^* t = m^* t$$

or, equivalently,

$$t(X_2, q_2) t(X_2, q_1) = t(X_2, m).$$

This last identity and the fact that t is a morphism of sheaves yields the commutativity of (5.4.1).

Conversely, if $(M, \tau) \in \mathsf{Desc}(A_\bullet)$, using (5.4.1) it is easy to prove that the following composition

$$p_2^* \widetilde{M} \xrightarrow{\sim} (A_1\, \widetilde{{}_{\eta_R}\!\otimes_{A_0}} M) \xrightarrow{\widetilde{\tau^{-1}}} (A_1\, \widetilde{{}_{\eta_L}\!\otimes_{A_0}} M) \xrightarrow{\sim} p_1^* \widetilde{M}$$

is a descent datum on $\widetilde{M}$. □

Theorem 5.8. *The categories* $\mathsf{Desc}_{\mathsf{qc}}(X/\mathbf{X})$ *and* $A_\bullet$-coMod *are equivalent.*

Proof. By Proposition 5.7 the category $\mathsf{Desc}_{\mathsf{qc}}(X/\mathbf{X})$ is equivalent to $\mathsf{Desc}(A_\bullet)$, which, in turn is equivalent to $A_\bullet$-coMod by Proposition 5.5. □

Remark. This result is similar to [Ho1, Theorem 2.2]. Notice however that Hovey's quasi-coherent sheaves over $\mathbf{X}$ are sheaves $\mathscr{F}$ for the (big) *discrete* topology while we work in the fppf topology. The ultimate reason of the agreement of both approaches lies in Lemma 3.6 that expresses the property of

descent of Cartesian presheaves for this topology which makes them automatically sheaves. Notice also that the present proof is simpler once we have made explicit in Proposition 5.7 that all the data involved refers to modules on an affine scheme and, therefore, it has an essentially algebraic nature.

Corollary 5.9. *For $\mathbf{X} = \mathsf{Stck}(A_\bullet)$, the categories $\mathsf{Qco}(\mathbf{X})$ and $A_\bullet$-coMod are equivalent.*

Proof. Combine Corollary 4.7 and Theorem 5.8. □

We denote by $\boldsymbol{\Gamma}^{\mathbf{X}}_p \colon \mathsf{Qco}(\mathbf{X}) \to A_\bullet\text{-}\mathsf{coMod}$ the equivalence of categories defined as $\boldsymbol{\Gamma}^{\mathbf{X}}_p := \mathsf{C} \circ \Gamma(X,-) \circ \mathsf{D}$, with D as in 4.2 and C as in Proposition 5.5. A quasi-inverse is $\Theta^{\mathbf{X}}_p = \mathsf{G} \circ \widetilde{(-)} \circ \mathsf{D}'$.

Corollary 5.10. *For $\mathbf{X} = \mathsf{Stck}(A_\bullet)$, $\mathsf{Qco}(\mathbf{X})$ is a Grothendieck category.*

Proof. Combine Theorem 5.3 with Corollary 5.9. □

Proposition 5.11. *We have*

(*i*) *If (M,ψ) is an $A_\bullet$-comodule and $\mathscr{F}$ is its associated $\mathscr{O}_{\mathbf{X}}$-module, then the $\widetilde{A_0}$-module $p^*\mathscr{F}$ is $\widetilde{M}$.*

(*ii*) *If $\mathscr{G} \in \mathsf{Qco}(X)$ then the comodule $\boldsymbol{\Gamma}^{\mathbf{X}}_p(p_*\mathscr{G})$ is isomorphic to the extended comodule $A_1\,{}_{\eta_R}\!\otimes_{A_0} \mathscr{G}(X)$.*

Proof. By Theorem 5.8 (M,ψ) corresponds to $(\widetilde{M},t) \in \mathsf{Desc}_{\mathsf{qc}}(X/\mathbf{X})$. By Corollary 5.9 $p^*\mathscr{F} = p^*\mathsf{G}(\widetilde{M},t) \cong \widetilde{M}$. This gives (*i*).

For (*ii*), we have by 2.2 that $\widetilde{\mathscr{G}(X)}$ and $\mathscr{G}$ are isomorphic $\widetilde{A_0}$-modules and then the comodules $\boldsymbol{\Gamma}^{\mathbf{X}}_p(p_*\widetilde{\mathscr{G}(X)}) = A_1\,{}_{\eta_R}\!\otimes_{A_0}\mathscr{G}(X)$ and $\boldsymbol{\Gamma}^{\mathbf{X}}_p(p_*\mathscr{G}) = \mathscr{G}(X_1,p_2)$ are isomorphic via $\mathscr{G}(p_2)^{\mathsf{a}} \colon A_1\,{}_{\eta_R}\!\otimes_{A_0}\mathscr{G}(X) \to \mathscr{G}(X_1,p_2)$. To prove (*ii*) is sufficient to show that the comodule structure on $A_1\,{}_{\eta_R}\!\otimes_{A_0}\mathscr{G}(X)$ given by 4.2, Proposition 5.7 and Proposition 5.5, is $\nabla \otimes \mathrm{id}$. Set $M = \mathscr{G}(X)$. The descent datum τ on $A_1\,{}_{\eta_R}\!\otimes_{A_0}M$ is defined by

$$\tau = (p_*\widetilde{M}(p_2,\mathsf{id})^{\mathsf{a}})^{-1} p_*\widetilde{M}(\mathsf{id},\phi^{-1})(p_*\widetilde{M}(p_1,\mathsf{id}))^{\mathsf{a}}.$$

The structure map ψ is the composition

$$A_1\,{}_{\eta_R}\!\otimes_{A_0}M \xrightarrow{\eta_L\otimes\mathrm{id}} A_1\,{}_{\eta_L}\!\otimes_{\eta_L}A_1\,{}_{\eta_R}\!\otimes_{A_0}M \xrightarrow{\tau} A_1\,{}_{\eta_R}\!\otimes_{\eta_L}A_1\,{}_{\eta_R}\!\otimes_{A_0}M.$$

Consider the isomorphism $w\colon X_2 \xrightarrow{\sim} X \times_{\mathbf{X}} X \times_{\mathbf{X}} X$ in (5.7.1). We obtain

$$\begin{aligned}
\psi &= (p_*\widetilde{M}(p_2,\mathsf{id})^{\mathsf{a}})^{-1} p_*\widetilde{M}(\mathsf{id},\phi^{-1}) p_*\widetilde{M}(p_1,\mathsf{id})\\
&= (p_*\widetilde{M}(p_2,\mathsf{id})^{\mathsf{a}})^{-1} p_*\widetilde{M}(p_1,\phi^{-1})\\
&= (p_*\widetilde{M}(p_2,\mathsf{id})^{\mathsf{a}})^{-1} \widetilde{M}(w)^{-1}\widetilde{M}(w)\widetilde{M}(p_{13})\\
&= \widetilde{M}(m) = \nabla \otimes \mathrm{id}.
\end{aligned}$$

the last equalities hold because $\widetilde{M}(w)p_*\widetilde{M}(p_2,\mathsf{id})^{\mathsf{a}} = \mathsf{id}$ and $m = \mathrm{Spec}(\nabla)$. □

6. Properties and functoriality of quasi-coherent sheaves

We have reached our first main objective: representing quasi-coherent sheaves on a geometric stack. It is time to move forward to the next one, namely, describing a nice functoriality formalism for these kind of a sheaves for an *arbitrary* 1-morphism of stacks. Let then $f\colon \mathbf{X} \to \mathbf{Y}$ be a 1-morphism of geometric stacks.

Unfortunately, f does not induce a (continuous) functor from $\mathsf{Aff}_{\mathsf{fppf}}/\mathbf{Y}$ to $\mathsf{Aff}_{\mathsf{fppf}}/\mathbf{X}$, so we can not apply directly the methods from 1.7 and 1.9. Of course, it could be done if f is an affine 1-morphism, but this condition is clearly too restrictive.

Besides, there is no hope to define a topos morphism from $\mathbf{X}_{\mathsf{fppf}}$ to $\mathbf{Y}_{\mathsf{fppf}}$ because the topology is too fine, however we will construct an adjunction that will induce the corresponding pair of adjoint functors between the categories of quasi-coherent sheaves.

6.1. Direct image. For fixed $(V,v) \in \mathsf{Aff}_{\mathsf{fppf}}/\mathbf{Y}$ we denote by $\mathbf{J}(v,f)$ the category whose objects are the 2-commutative squares

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[d, "u"'] & V \arrow[d, "v"] \\ \mathbf{X} \arrow[r, "f"'] \arrow[ur, Rightarrow, "\gamma"] & \mathbf{Y} \end{tikzcd} \tag{6.1.1}$$

where $(U,u) \in \mathsf{Aff}_{\mathsf{fppf}}/\mathbf{X}$. We denote this object by (U,u,h,γ). A morphism from (U,u,h,γ) to (U',u',h',γ') is just a morphism $(l,\alpha)\colon (U,u) \to (U',u')$ in $\mathsf{Aff}_{\mathsf{fppf}}/\mathbf{X}$ verifying $h'l = h$ and $\gamma = \gamma' l \circ f\alpha$. If $\mathscr{G} \in \mathsf{Pre}(\mathsf{Aff}_{\mathsf{fppf}}/\mathbf{X})$ we define the presheaf $f_*\mathscr{G}$ by:

$$(f_*\mathscr{G})(V,v) = \varprojlim_{\mathbf{J}(v,f)} \mathscr{G}(U,u).$$

It is a presheaf. Let (V_1,v_1), $(V_2,v_2) \in \mathsf{Aff}_{\mathsf{fppf}}/\mathbf{Y}$ and $(j,\beta)\colon (V_1,v_1) \to (V_2,v_2)$. In this situation we define a functor

$$(j,\beta)_*\colon \mathbf{J}(v_1,f) \longrightarrow \mathbf{J}(v_2,f)$$

by $(j,\beta)_*(U,u,h,\gamma) := (U,u,jh,\beta h \circ \gamma)$. This induces the restriction morphism

$$(f_*\mathscr{G})(V_2,v_2) \longrightarrow (f_*\mathscr{G})(V_1,v_1).$$

Notice that if f is an affine morphism, then $f_*\mathscr{G}(V,v) = \mathscr{G}(\mathbf{X}\times_{\mathbf{Y}} V, p_1)$ because $(\mathbf{X}\times_{\mathbf{Y}}, p_1, p_2, \phi_0)$, with ϕ_0 the canonical 2-cell in the 2-pullback, is an final object of $\mathbf{J}(v,f)$.

6.2. Inverse image. Let $(U,u) \in \mathsf{Aff}_{\mathsf{fppf}}/\mathbf{X}$, we denote by $\mathbf{I}(u,f)$ the category whose objects are the 2-commutative squares as 6.1.1 with $(V,v) \in \mathsf{Aff}_{\mathsf{fppf}}/\mathbf{Y}$. We denote these objects now as (V,v,h,γ). A morphism from (V,v,h,γ) to

(V',v',h',γ') is a morphism $(j,\beta)\colon (V,v)\to(V',v')$ in $\mathsf{Aff}_{\mathsf{fppf}}/\mathbf{Y}$ verifying $jh=h'$ and $\gamma'=\beta h\circ\gamma$. For $\mathscr{F}\in\mathsf{Pre}(\mathsf{Aff}_{\mathsf{fppf}}/\mathbf{Y})$, the presheaf $f^{\mathsf{p}}\mathscr{F}$ is given by

$$(f^{\mathsf{p}}\mathscr{F})(U,u)=\varinjlim_{\mathbf{I}(u,f)}\mathscr{F}(V,v)$$

for each $(U,u)\in\mathsf{Aff}_{\mathsf{fppf}}/\mathbf{X}$. It is also a presheaf. Consider (U_1,u_1), $(U_2,u_2)\in\mathsf{Aff}_{\mathsf{fppf}}/\mathbf{X}$ together with $(l,\alpha)\colon (U_1,u_1)\to(U_2,u_2)$. In this situation we define a functor

$$(l,\alpha)^*\colon \mathbf{I}(u_2,f)\longrightarrow\mathbf{I}(u_1,f)$$

by $(l,\alpha)^*(V,v,h,\gamma):=(V,v,hl,\gamma l\circ f\alpha)$. This induces the restriction morphism

$$(f^{\mathsf{p}}\mathscr{F})(U_2,u_2)\longrightarrow(f^{\mathsf{p}}\mathscr{F})(U_1,u_1).$$

If f is a flat finitely presented morphism, then, $f^{\mathsf{p}}\mathscr{F}(U,u)=\mathscr{F}(U,fu)$ because $(U,fu,\mathrm{id}_U,\mathrm{id})$ is an initial object of $\mathbf{I}(u,f)$.

Lemma 6.3. *Let $f_1,f_2\colon \mathbf{X}\to\mathbf{Y}$ be 1-morphisms of geometric stacks. In the previous setting, any 2-morphism $\zeta\colon f_1\Rightarrow f_2$ induces equivalences of categories*

(*i*) $\zeta_{\mathbf{J}}\colon \mathbf{J}(v,f_1)\xrightarrow{\sim}\mathbf{J}(v,f_2)$,
(*ii*) $\zeta^{\mathbf{I}}\colon \mathbf{I}(u,f_2)\xrightarrow{\sim}\mathbf{I}(u,f_1)$.

Proof. For (*i*), define, using the notation as in 6.1,

$$\zeta_{\mathbf{J}}(U,u,h,\gamma)=(U,u,h,\gamma\circ\zeta^{-1}u)$$

and extend in an obvious way to maps. We see that these data defines a functor and a quasi-inverse is given by $(\zeta^{-1})_{\mathbf{J}}$.

Analogously, for (*ii*), define, using the notation as in 6.2,

$$\zeta^{\mathbf{I}}(V,v,h,\gamma)=(V,v,h,\gamma\circ\zeta u)$$

As before, this defines a functor whose quasi-inverse is $(\zeta^{-1})^{\mathbf{I}}$. □

Proposition 6.4. *Let $f_1,f_2\colon \mathbf{X}\to\mathbf{Y}$ be 1-morphisms of geometric stacks. With the previous notation, given a 2-morphism $\zeta\colon f_1\Rightarrow f_2$ we have the following isomorphisms*

(*i*) $\zeta_*\colon f_{1*}\xrightarrow{\sim}f_{2*}$,
(*ii*) $\zeta^{\mathsf{p}}\colon f_2^{\mathsf{p}}\xrightarrow{\sim}f_1^{\mathsf{p}}$.

Proof. To prove (*i*), apply the equivalence $\zeta_{\mathbf{J}}\colon \mathbf{J}(v,f_1)\xrightarrow{\sim}\mathbf{J}(v,f_2)$ to the sets $\mathscr{G}(U,u)$ with $(U,u)\in\mathbf{J}(v,f_1)$, this gives a map

$$\varprojlim_{\mathbf{J}(v,f_1)}\mathscr{G}(U,u)\xrightarrow{\sim}\varprojlim_{\mathbf{J}(v,f_2)}\mathscr{G}(U,u).$$

This isomorphism is natural, therefore induces a natural isomorphism

$$f_{1*}\mathscr{G}\xrightarrow{\sim}f_{2*}\mathscr{G}.$$

For (*ii*) use a similar argument with $\zeta^{\mathbf{I}}$ in place of $\zeta_{\mathbf{J}}$. □

Proposition 6.5. *The previous constructions define a pair of adjoint functors*

$$\mathsf{Pre}(\mathsf{Aff}_{\mathsf{fppf}}/\mathbf{X})\underset{f_*}{\overset{f^{\mathsf{p}}}{\leftrightarrows}}\mathsf{Pre}(\mathsf{Aff}_{\mathsf{fppf}}/\mathbf{Y}).$$

Proof. We have to prove, for $\mathscr{F} \in \mathsf{Pre}(\mathsf{Aff}_{\mathsf{fppf}}/\mathbf{Y})$ and $\mathscr{G} \in \mathsf{Pre}(\mathsf{Aff}_{\mathsf{fppf}}/\mathbf{X})$ that there is an isomorphism

$$\mathrm{Hom}_{\mathbf{X}}(f^{\mathsf{p}}\mathscr{F},\mathscr{G}) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{Y}}(\mathscr{F}, f_*\mathscr{G})$$

A map $f^{\mathsf{p}}\mathscr{F} \to \mathscr{G}$ induces for every $(U,u) \in \mathsf{Aff}_{\mathsf{fppf}}/\mathbf{X}$ and $(V,v) \in \mathbf{I}(u,f)$ morphisms

$$\mathscr{F}(V,v) \longrightarrow \varinjlim_{\mathbf{I}(u,f)} \mathscr{F}(V,v) \longrightarrow \mathscr{G}(U,u)$$

Now fixing (V,v) and varying (U,u) we get a map

$$\mathscr{F}(V,v) \longrightarrow \varprojlim_{\mathbf{J}(v,f)} \mathscr{G}(U,u).$$

This construction provides the adjunction map. By a similar procedure, we obtain a map in the opposite direction. It is not difficult to check that both maps are mutually inverse. □

6.6. The *continuity* of our construction is expressed by the relation between the coverings. Let us spell it out.

Take a covering $\{(g_i,\beta_i)\colon (V_i,v_i) \to (V,v)\}_{i\in I}$ in the site $\mathsf{Aff}_{\mathsf{fppf}}/\mathbf{Y}$. Given $(U,u,h,\gamma) \in \mathbf{J}(v,f)$, consider for each $i \in I$ the following Cartesian square of affine schemes

$$\begin{tikzcd}
U \times_V V_i \arrow[r, "h_i"] \arrow[d, "g'_i"'] & V_i \arrow[d, "g_i"] \\
U \arrow[r, "h"] & V
\end{tikzcd}$$

Set $U_i := U \times_V V_i$ and $\gamma_i\colon fug'_i \Rightarrow v_i h_i$, given by $\gamma_i := \beta_i^{-1} h_i \circ \gamma g'_i$. This defines a functor

$$\overline{g_i}\colon \mathbf{J}(v,f) \longrightarrow \mathbf{J}(v_i,f) \tag{6.6.1}$$

by $\overline{g_i}(U,u,h,\gamma) := (U_i, ug'_i, h_i, \gamma_i)$.

A consequence of this observation is the following important statement.

Proposition 6.7. *If $\mathscr{G}$ is a sheaf on $\mathsf{Aff}_{\mathsf{fppf}}/\mathbf{X}$, then $f_*\mathscr{G}$ is a sheaf.*

Proof. Let $\{(g_i,\beta_i)\colon (V_i,v_i) \to (V,v)\}_{i\in I}$ be a covering in the site $\mathsf{Aff}_{\mathsf{fppf}}/\mathbf{Y}$. With the previous notation, given $(U,u,h,\gamma) \in \mathbf{J}(v,f)$, the family

$$\{(g'_i,\mathsf{id})\colon (U_i, ug'_i) \to (U,u)\}_{i\in I}$$

is a covering in the site $\mathsf{Aff}_{\mathsf{fppf}}/\mathbf{X}$. Associated to the sheaf $\mathscr{G}$ we have an equalizer diagram

$$\mathscr{G}(U) \longrightarrow \textstyle\prod_{i\in I} \mathscr{G}(U_i) \overset{\rho_2}{\underset{\rho_1}{\rightrightarrows}} \textstyle\prod_{i,j\in I} \mathscr{G}(U_i \times_U U_j).$$

Taking limits, we obtain another equalizer diagram (in the following we will keep the notation ρ on the maps to avoid the clutter)

$$\varprojlim_{\mathbf{J}(v,f)} \mathscr{G}(U) \longrightarrow \varprojlim_{\mathbf{J}(v,f)} \prod_{i\in I} \mathscr{G}(U_i) \overset{\rho_2}{\underset{\rho_1}{\rightrightarrows}} \varprojlim_{\mathbf{J}(v,f)} \prod_{i,j\in I} \mathscr{G}(U_i \times_U U_j).$$

That, by commutation of limits [Bo, Prop. 2.12.1], it may be rewritten as

$$\varprojlim_{\mathbf{J}(v,f)} \mathscr{G}(U) \longrightarrow \prod_{i\in I} \varprojlim_{\mathbf{J}(v,f)} \mathscr{G}(U_i) \overset{\rho_2}{\underset{\rho_1}{\rightrightarrows}} \prod_{i,j\in I} \varprojlim_{\mathbf{J}(v,f)} \mathscr{G}(U_i \times_U U_j).$$

Let us denote by $\mathbf{J}_i(v,f)$ the essential image of $\mathbf{J}(v,f)$ in $\mathbf{J}(v_i,f)$ through the functor $\overline{g_i}$ (6.6.1). Note that

$$\varprojlim_{\mathbf{J}(v,f)} \mathscr{G}(U_i) \quad = \quad \varprojlim_{\mathbf{J}_i(v,f)} \mathscr{G}(U_i)$$

All of this applies verbatim to $\mathbf{J}_{ij}(v,f)$.

Consider now the following commutative diagram

$$\begin{tikzcd}
\varprojlim_{\mathbf{J}(v,f)} \mathscr{G}(U) \arrow[r] & \prod_{i\in I} \varprojlim_{\mathbf{J}_i(v,f)} \mathscr{G}(U_i) \arrow[r, shift left, "\rho_2"] \arrow[r, shift right, "\rho_1"'] & \prod_{i,j\in I} \varprojlim_{\mathbf{J}_{ij}(v,f)} \mathscr{G}(U_i \times_U U_j) \\
f_*\mathscr{G}(V) \arrow[u, equal] \arrow[r] & \prod_{i\in I} f_*\mathscr{G}(V_i) \arrow[u, "\mathrm{can}"'] \arrow[r, shift left, "\rho_2"] \arrow[r, shift right, "\rho_1"'] & \prod_{i,j\in I} f_*\mathscr{G}(V_i \times_V V_j) \arrow[u, "\mathrm{can}"']
\end{tikzcd}$$

In a natural way, for every $\overline{U} = (U,u,h,\gamma) \in \mathbf{J}(v_i,f)$ we have $(U,u,g_1h,\beta h\circ\gamma) \in \mathbf{J}(v,f)$. Then we get

$$\overline{U}_i = \overline{g_i}(U,u,g_1h,\beta h\circ\gamma) \in \mathbf{J}_i(v,f)$$

jointly with maps in $\mathbf{J}(v_i,f)$, $\overline{U} \to \overline{U}_i$ and $\overline{U}_i \to \overline{U}$ whose composition is the identity. This implies that the canonical map

$$f_*\mathscr{G}(V_i) \quad \xrightarrow{\mathrm{can}} \quad \varprojlim_{\mathbf{J}_i(v,f)} \mathscr{G}(U_i)$$

is a monomorphism. Similarly the rightmost vertical map is an monomorphism, too. The top row is an equalizer and this implies that the bottom row is, as a consequence, $f_*\mathscr{G}$ a sheaf. □

Corollary 6.8. *Let $\mathscr{F} \in \mathsf{Y}_{\mathsf{fppf}}$. We denote by $f^{-1}\mathscr{F}$ the sheaf associated to the presheaf $f^{\mathsf{p}}\mathscr{F}$. There is a pair of adjoint functors*

$$\mathsf{X}_{\mathsf{fppf}} \underset{f_*}{\overset{f^{-1}}{\leftrightarrows}} \mathsf{Y}_{\mathsf{fppf}}.$$

Proof. Combine the previous Propositions 6.5 and 6.7, having in mind the adjoint property of sheafification. □

Remark. Notice, however, that, in general, the pair (f^{-1}, f_*) *does not* define a morphism of topoi, since f^{-1} need not be left exact. For an explicit example see [Be, 4.4.2]. In the case in which f is a flat finitely presented morphism, then f^{-1} is exact.

Proposition 6.9. *Let* $f\colon \mathsf{X} \to \mathsf{Y}$ *and* $g\colon \mathsf{Y} \to \mathsf{Z}$ *be 1-morphisms of geometric stacks. It holds that* $(gf)_* \cong g_* f_*$

Proof. Let $\mathscr{F} \in \mathsf{X}_{\mathsf{fppf}}$. For each $(W,w) \in \mathsf{Aff}_{\mathsf{fppf}}/\mathsf{Z}$, we have that

$$(gf)_*\mathscr{F}(W,w) \quad = \quad \varprojlim_{\mathbf{J}(w,gf)} \mathscr{F}(U,u)$$

where $\mathbf{J}(w,gf)$ is as in 6.1 for (W,w). On the other hand

$$g_* f_* \mathscr{F}(W,w) \quad = \quad \varprojlim_{\mathbf{J}(w,g)} f_*\mathscr{F}(V,v)$$

We have a family of maps compatible with morphisms in $\mathbf{J}(w,g)$

$$(gf)_*\mathscr{F}(W,w) \longrightarrow f_*\mathscr{F}(V,v)$$

defining a natural map

$$(gf)_*\mathscr{F}(W,w) \longrightarrow g_* f_* \mathscr{F}(W,w). \tag{6.9.1}$$

Let us show that this map is an isomorphism.

Next, for each $(U,u,h,\gamma) \in \mathbf{J}(w,gf)$ we will construct a map

$$g_* f_* \mathscr{F}(W,w) \longrightarrow \mathscr{F}(U,u)$$

compatible with morphisms in $\mathbf{J}(w,gf)$. Consider the 2-fibered product stack $\mathsf{Y}' := \mathsf{Y} \times_{\mathsf{Z}} W$, its canonical projections q_1 and q_2, and $\gamma_0\colon gq_1 \Rightarrow wq_2$ its associated 2-morphism. Y' is a geometric stack by Proposition 3.3. There is a morphism $f'\colon U \to \mathsf{Y}'$ given by the 2-isomorphism $\gamma\colon gfu \Rightarrow wh$, such that $q_1 f' = fu$, $h = q_2 f'$ and $\gamma_0 f' = \gamma$. Let $q\colon V' \to \mathsf{Y}'$ be a presentation. Consider the 2-fiber squares

$$\begin{tikzcd} U' \arrow[r, "f''"] \arrow[d, "q'"'] & V' \arrow[d, "q"] \\ U \arrow[r, "f'"'] \arrow[ur, phantom, "\gamma''"] & \mathsf{Y}' \end{tikzcd} \qquad \begin{tikzcd} V'' \arrow[r, "p'_2"] \arrow[d, "p'_1"'] & V' \arrow[d, "q"] \\ V' \arrow[r, "q"'] \arrow[ur, phantom, "\gamma'''"] & \mathsf{Y}' \end{tikzcd}$$

Denote by $v'\colon V' \to \mathsf{Y}$ the map $v' = q_1 q$. It is clear that $(V', v', q_2 q, \gamma_0 q) \in \mathbf{J}(w,g)$. Take $v'' := v' p'_1$. In $\mathbf{J}(w,g)$, we have the projections (p'_1, id) and $(p'_2, q_1\gamma''')$ from $(V'', v'', q_2 q p'_1, \gamma_0 q p'_1)$ to $(V', v', q_2 q, \gamma_0 q)$. We have a diagram

$$\begin{tikzcd} g_* f_* \mathscr{F}(W,w) \arrow[r, "\lambda"] \arrow[d, dashed] & f_*\mathscr{F}(V',v') \arrow[r, shift left, "f_*\mathscr{F}(p'_1,\mathsf{id})"] \arrow[r, shift right, "f_*\mathscr{F}(p'_2,q_1\gamma''')"'] \arrow[d, "\mathsf{can}"] & f_*\mathscr{F}(V'',v'') \arrow[d, "\mathsf{can}"] \\ \mathscr{F}(U,u) \arrow[r, "\mathscr{F}(q',\mathsf{id})"] & \mathscr{F}(U',u') \arrow[r, shift left, "\mathscr{F}(p_1,\mathsf{id})"] \arrow[r, shift right, "\mathscr{F}(p_2,\mathsf{id})"'] & \mathscr{F}(U' \times_U U', u'') \end{tikzcd}$$

where $p_1, p_2 \colon U' \times_U U' \to U'$ are the canonical projections, $u' = uq'$, $u'' = u'p_1$ and the vertical maps labelled "can" are induced by the limit diagrams indexed by $\mathbf{J}(v', f)$ and $\mathbf{J}(v'', f)$. The map λ is the canonical map corresponding to the limit of the diagram indexed by $\mathbf{J}(w, g)$.

Notice that q is smooth and surjective and the same holds for q' therefore, $\{q' \colon U' \to U\}$ is a (flat) covering in $\mathsf{Aff}_{\mathsf{fppf}}/\mathbf{X}$. As $\mathscr{F}$ is a sheaf, the lower row is an equalizer diagram. We have that $f_*\mathscr{F}(p'_1, \mathrm{id})\lambda = f_*\mathscr{F}(p'_2, q_1\gamma''')\lambda$, therefore there is a canonical vertical dashed map. Passing to the limit along $\mathbf{J}(w, gf)$ we obtain a natural map

$$g_* f_* \mathscr{F}(\mathrm{W}, w) \longrightarrow (gf)_* \mathscr{F}(\mathrm{W}, w)$$

which is inverse to 6.9.1. $\square$

Corollary 6.10. *In the previous situation* $(gf)^{-1} \xrightarrow{\sim} f^{-1}g^{-1}$.

Proof. It is is a consequence of the previous proposition by adjunction. $\square$

6.11. In general, for $f \colon \mathbf{X} \to \mathbf{Y}$, there is no map between ringed sites, however, we consider the morphism of sheaves of rings $f^\# \colon \mathscr{O}_{\mathbf{Y}} \to f_*\mathscr{O}_{\mathbf{X}}$, given for each $(V, v) \in \mathsf{Aff}_{\mathsf{fppf}}/\mathbf{Y}$ by the induced ring homomorphism

$$f^\#_{(V,v)} \colon \mathscr{O}_{\mathbf{Y}}(V, v) \longrightarrow \varprojlim_{\mathbf{J}(v,f)} \mathscr{O}_{\mathbf{X}}(U, u).$$

We want to extend the previous adjunction to the categories of sheaves of modules. As we do not have neither a topos morphism neither a continuous map of sites, this extension is not straightforward.

The functor f_* takes $\mathscr{O}_{\mathbf{X}}$-Modules to $f_*\mathscr{O}_{\mathbf{X}}$-Modules and, by the forgetful functor associated to $f^\#$, to $\mathscr{O}_{\mathbf{Y}}$-Modules. Note that the formula in Proposition 6.9 keeps holding by the transitivity of the forgetful functors.

We will apply Lemma 1.8 to guarantee that the functor f^{p} commutes with finite products and so does f^{-1}. It will follow that if $\mathscr{F}$ is an $\mathscr{O}_{\mathbf{Y}}$-module, $f^{-1}\mathscr{F}$ will be an $f^{-1}\mathscr{O}_{\mathbf{Y}}$-module. Therefore we have to check the hypothesis of lemma 1.8, so we are reduced to prove the following.

Lemma 6.12. *The category* $\mathbf{I}(u, f)$ *has finite products.*

Proof. We only need to prove that any two objects in $\mathbf{I}(u, f)$ admit a product. For that, let $(V_1, v_1, h_1, \gamma_1)$ and $(V_2, v_2, h_2, \gamma_2)$ be objects in $\mathbf{I}(u, f)$. For $i \in \{1, 2\}$, let p_i be the projection from $V_1 \times_{\mathbf{Y}} V_2$ to V_i, and $\beta \colon v_1p_1 \Rightarrow v_2p_2$ be the canonical 2-morphism. If $h \colon U \to V_1 \times_{\mathbf{Y}} V_2$ is the morphism verifying $p_1h = h_1$, $p_2h = h_2$ and $\beta h = \gamma_2 \circ \gamma_1^{-1}$, then the object $(V_1 \times_{\mathbf{Y}} V_2, v_1p_1, h, \gamma_1)$ together with the morphisms (p_1, id) and (p_2, β) is a product of $(V_1, v_1, h_1, \gamma_1)$ and $(V_2, v_2, h_2, \gamma_2)$ in $\mathbf{I}(u, f)$. $\square$

Corollary 6.13. *The functor* $f^* \colon \mathscr{O}_{\mathbf{Y}}\text{-}\mathsf{Mod} \to \mathscr{O}_{\mathbf{X}}\text{-}\mathsf{Mod}$ *given by*

$$f^*\mathscr{F} := \mathscr{O}_{\mathbf{X}} \otimes_{f^{-1}\mathscr{O}_{\mathbf{Y}}} f^{-1}\mathscr{F}$$

is left adjoint to the functor $f_* \colon \mathscr{O}_{\mathbf{X}}\text{-}\mathsf{Mod} \to \mathscr{O}_{\mathbf{Y}}\text{-}\mathsf{Mod}$.

Proof. Combine Corollary 6.8 with the previous discussion. $\square$

Corollary 6.14. *Let $f : \mathbf{X} \to \mathbf{Y}$ and let $g : \mathbf{Y} \to \mathbf{Z}$ be 1-morphisms of geometric stacks. It holds that $(gf)^* \cong f^* g^*$.*

Proof. It is a consequence of Proposition 6.9 by adjunction. □

Corollary 6.15. *Let $f_1, f_2 : \mathbf{X} \to \mathbf{Y}$ 1-morphisms of geometric stacks. With the previous notation, given a 2-morphism $\zeta : f_1 \Rightarrow f_2$ we have isomorphisms*

(*i*) $\zeta_* : f_{1*} \xrightarrow{\sim} f_{2*}$
(*ii*) $\zeta^* : f_2^* \xrightarrow{\sim} f_1^*$

as functors defined on $\mathcal{O}_{\mathbf{X}}$-Mod and $\mathcal{O}_{\mathbf{Y}}$-Mod, respectively.

Proof. Immediate from 6.4 and the previous discussion. □

Remark. Notice that the isomorphism ϕ^* in 4.2 is a particular case of (*ii*).

Proposition 6.16. *If $\mathcal{F} \in \mathsf{Qco}(\mathbf{Y})$ then $f^*\mathcal{F} \in \mathsf{Qco}(\mathbf{X})$.*

Proof. Let $p : X \to \mathbf{X}$ and $q : Y \to \mathbf{Y}$ be affine presentations such that we have a 2-commutative square

$$\begin{tikzcd} X \arrow[r, "f_0"] \arrow[d, "p"'] & Y \arrow[d, "q"] \\ \mathbf{X} \arrow[r, "f"'] \arrow[ur, Rightarrow, "\gamma"] & \mathbf{Y} \end{tikzcd}$$

It follows from the previous Corollary that $p^* f^* \mathcal{F} \cong f_0^* q^* \mathcal{F}$, via γ^* using pseudofunctoriality. By Proposition 3.8 $q^*\mathcal{F} \in \mathsf{Crt}(Y)$ and by Proposition 2.4, $f_0^* q^* \mathcal{F} \in \mathsf{Crt}(X)$. We conclude applying Lemma 3.10 and Theorem 3.11. □

Proposition 6.17. *If $\mathcal{G} \in \mathsf{Qco}(\mathbf{X})$ then $f_*\mathcal{G} \in \mathsf{Qco}(\mathbf{Y})$.*

Proof. If f is an affine morphism the proof is similar to the given in Proposition 2.4. If f is not affine, take an affine presentation $p : X \to \mathbf{X}$. The morphism fp is affine. Let $\mathsf{T} := p_* p^*$ as in Lemma 3.10 and consider the following equalizer of quasi-coherent sheaves

$$\mathcal{G} \xrightarrow{\eta_{\mathcal{G}}} \mathsf{T}\mathcal{G} \underset{\eta_{\mathsf{T}\mathcal{G}}}{\overset{\mathsf{T}\eta_{\mathcal{G}}}{\rightrightarrows}} \mathsf{T}^2\mathcal{G}.$$

Since f_* is left exact we have the equalizer of $\mathcal{O}_{\mathbf{Y}}$-modules

$$f_*\mathcal{G} \xrightarrow{f_*\eta_{\mathcal{G}}} f_*\mathsf{T}\mathcal{G} \underset{f_*\eta_{\mathsf{T}\mathcal{G}}}{\overset{f_*\mathsf{T}\eta_{\mathcal{G}}}{\rightrightarrows}} f_*\mathsf{T}^2\mathcal{G}.$$

From Proposition 3.8 and Theorem 3.11 by using Proposition 6.9 it follows that $f_*\mathsf{T}\mathcal{G}$ and $f_*\mathsf{T}^2\mathcal{G}$ are quasi-coherent sheaves. Then, $f_*\mathcal{G}$ is a quasi-coherent Module since it is the kernel of a morphism of quasi-coherent Modules. □

Corollary 6.18. *There is a pair of adjoint functors*

$$\mathsf{Qco}(\mathbf{X}) \underset{f_*}{\overset{f^*}{\leftrightarrows}} \mathsf{Qco}(\mathbf{Y}).$$

Proof. Combine the adjunction of Corollary 6.13 with Propositions 6.16 and 6.17. □

7. Describing functoriality via comodules

Let $\varphi\colon A_\bullet \to B_\bullet$ be a homomorphism of Hopf algebroids. Following [Ho2], we describe a pair of adjoint functors

$$B_\bullet\text{-}\mathsf{coMod} \underset{\mathsf{U}^\varphi}{\overset{B_0\otimes_{A_0}-}{\longleftrightarrow}} A_\bullet\text{-}\mathsf{coMod}.$$

We will see that for the corresponding 1-morphism of stacks $f := \mathsf{Stck}(\varphi)$

$$f\colon \mathsf{Stck}(B_\bullet) \longrightarrow \mathsf{Stck}(A_\bullet),$$

this adjunction corresponds to the previously considered $f^* \dashv f_*$ between sheaves of quasi-coherent modules on the small flat site.

7.1. A homomorphism of Hopf algebroids $\varphi\colon A_\bullet \to B_\bullet$ is a pair of ring homomorphisms $\varphi_i\colon A_i \to B_i$ with $i \in \{0,1\}$ that respects the structure of a Hopf algebroid, namely, the following equalities hold

$$\begin{array}{lll} \varphi_1\eta_L = \eta_L\varphi_0 & \epsilon\varphi_1 = \varphi_0\epsilon & \\ \varphi_1\eta_R = \eta_R\varphi_0 & \kappa\varphi_1 = \varphi_1\kappa & (\varphi_1\otimes\varphi_1)\nabla = \nabla\varphi_1. \end{array}$$

7.2. The homomorphism $\varphi\colon A_\bullet \to B_\bullet$ induces a functor

$$B_0\otimes_{A_0} -\colon A_\bullet\text{-}\mathsf{coMod} \longrightarrow B_\bullet\text{-}\mathsf{coMod},$$

sending an $A_\bullet$-comodule (M,ψ) to the $B_\bullet$-comodule $(B_0\otimes_{A_0} M,\psi')$ where ψ' is given by the composition

$$B_0\otimes_{A_0} M \xrightarrow{\mathrm{id}\otimes\psi} B_0\otimes_{\eta_L} A_{1\,\eta_R}\otimes_{A_0} M \xrightarrow{\overline{\varphi}\otimes\mathrm{id}} B_{1\,\eta_R}\otimes_{A_0} M \xrightarrow{\sim} B_{1\,\eta_R}\otimes_{B_0}(B_0\otimes_{A_0} M)$$

where $\overline{\varphi}(b_0\otimes a_1) := \eta_L(b_0)\varphi_1(a_1)$, for $a_1\in A_1$ and $b_0\in B_0$.

7.3. From [Ho2, Proposition 1.2.3], we will describe

$$\mathsf{U}^\varphi\colon B_\bullet\text{-}\mathsf{coMod} \longrightarrow A_\bullet\text{-}\mathsf{coMod},$$

a right adjoint for the functor $B_0\otimes_{A_0} -$. This will be done in two steps.

Consider first an extended comodule, that is, given $N\in B_0$-Mod, let us consider the comodule $B_1\otimes_{B_0} N$ with the induced structure. We define

$$\mathsf{U}^\varphi(B_1\otimes_{B_0} N) := A_1\otimes_{A_0} N$$

with the induced structure. For maps $\lambda\colon B_1\otimes_{B_0} N \to B_1\otimes_{B_0} N'$ with $N,N'\in B_0$-Mod, then $\mathsf{U}^\varphi(\lambda)$ is defined as the following composition

$$\begin{aligned} A_{1\,\eta_R}\otimes_{A_0} N &\xrightarrow{\nabla\otimes\mathrm{id}} A_{1\,\eta_R}\otimes_{\eta_L} A_{1\,\eta_R}\otimes_{A_0} N \\ &\xrightarrow{\mathrm{id}\otimes\varphi_1\otimes\mathrm{id}} A_{1\,\eta_R}\otimes_{\eta_L\varphi_0} B_{1\,\eta_R}\otimes_{B_0} N \\ &\xrightarrow{\mathrm{id}\otimes\lambda} A_{1\,\eta_R}\otimes_{\eta_L\varphi_0} B_{1\,\eta_R}\otimes_{B_0} N' \\ &\xrightarrow{\mathrm{id}\otimes\epsilon\otimes\mathrm{id}} A_{1\,\eta_R}\otimes_{A_0} N'. \end{aligned}$$

Now, let (N,ψ) be an arbitrary $B_\bullet$-comodule. Let $\lambda'\colon B_1 \otimes_{B_0} N \to N'$ be the cokernel of ψ as B_0-modules. Let ψ' be the induced comodule structure on N' and consider the following $A_\bullet$-comodule homomorphism:

$$A_1\,{}_{\eta_R}\!\otimes_{A_0} N \xrightarrow{\mathsf{U}^\varphi(\psi'\lambda')} A_1\,{}_{\eta_R}\!\otimes_{A_0} N'.$$

Define

$$\mathsf{U}^\varphi(N) := \ker(\mathsf{U}^\varphi(\psi'\lambda'))$$

The extension to morphisms is straightforward by the naturally of kernels.

Proposition 7.4. *Given a homomorphism $\varphi\colon A_\bullet \to B_\bullet$ of Hopf algebroids, there is an associated pair of adjoint functors*

$$B_\bullet\text{-coMod} \underset{\mathsf{U}^\varphi}{\overset{B_0 \otimes_{A_0} -}{\leftrightarrows}} A_\bullet\text{-coMod}.$$

Proof. Let $(P,\psi) \in B_\bullet$-coMod and $(M,\psi') \in A_\bullet$-coMod. We have to construct an isomorphism

$$\mathrm{Hom}_{B_\bullet\text{-coMod}}(B_0 \otimes_{A_0} M, P) \xrightarrow{\sim} \mathrm{Hom}_{A_\bullet\text{-coMod}}(M, \mathsf{U}^\varphi(P))$$

First, if P is extended, *i.e.* $P = B_1 \otimes_{B_0} N$ for an B_0-module N, then we have that

$$\begin{aligned}
\mathrm{Hom}_{B_\bullet\text{-coMod}}(B_0 \otimes_{A_0} M, P) &= \mathrm{Hom}_{B_\bullet\text{-coMod}}(B_0 \otimes_{A_0} M, B_1 \otimes_{B_0} N)\\
&\cong \mathrm{Hom}_{B_0\text{-Mod}}(B_0 \otimes_{A_0} M, N)\\
&\cong \mathrm{Hom}_{A_0\text{-Mod}}(M, N)\\
&\cong \mathrm{Hom}_{A_\bullet\text{-coMod}}(M, A_1 \otimes_{A_0} N)\\
&= \mathrm{Hom}_{A_\bullet\text{-coMod}}(M, \mathsf{U}^\varphi(P))
\end{aligned}$$

In the general case, P may be obtained as a kernel of a morphism between extended $B_\bullet$-comodules, in other words, we have an exact sequence

$$0 \longrightarrow P \longrightarrow P' \longrightarrow P''$$

with P' and P'' extended $B_\bullet$-comodules. Now we have a diagram

$$\begin{tikzcd}
\mathrm{Hom}_{B_\bullet}(B_0 \otimes_{A_0} M, P) \arrow[r] \arrow[d, dashed] & \mathrm{Hom}_{B_\bullet}(B_0 \otimes_{A_0} M, P') \arrow[r] \arrow[d, "\wr"] & \mathrm{Hom}_{B_\bullet}(B_0 \otimes_{A_0} M, P'') \arrow[d, "\wr"] \\
\mathrm{Hom}_{A_\bullet}(M, \mathsf{U}^\varphi(P)) \arrow[r] & \mathrm{Hom}_{A_\bullet}(M, \mathsf{U}^\varphi(P')) \arrow[r] & \mathrm{Hom}_{A_\bullet}(M, \mathsf{U}^\varphi(P''))
\end{tikzcd}$$

in which the last two vertical maps are isomorphisms because P' and P'' are extended comodules, therefore they induce the dashed vertical map which is also an isomorphism as desired. $\square$

Remark. The previous result is due to Hovey [Ho2]. We have included a proof for the reader's convenience.

7.5. Our next task is to identify the functor U^{φ} with the direct image of sheaves on the geometric stacks corresponding to the Hopf algebroids.

Let $f\colon \mathbf{X} \to \mathbf{Y}$ be a morphism of geometric stacks, $p\colon X \to \mathbf{X}$ and $q\colon Y \to \mathbf{Y}$ affine presentations and $f_0\colon X \to Y$ be a morphism such that the following square

$$\begin{tikzcd} X \arrow[r, "f_0"] \arrow[d, "p"'] & Y \arrow[d, "q"] \\ \mathbf{X} \arrow[r, "f"'] \arrow[ur, Rightarrow, "\gamma"] & \mathbf{Y} \end{tikzcd}$$

is 2-commutative. Set $\mathbf{X} = \mathsf{Stck}(B_\bullet)$ and $\mathbf{Y} = \mathsf{Stck}(A_\bullet)$. Denote the ring homomorphisms associated to f as $\varphi_0\colon A_0 \to B_0$ and $\varphi_1\colon A_1 \to B_1$. We remind the reader that $\varphi = (\varphi_0, \varphi_1)$ is a homomorphism of Hopf algebroids (see 7.1).

Proposition 7.6. *There is a natural isomorphism of functors from* $\mathsf{Qco}(\mathbf{X})$ *to* $A_\bullet$-coMod

$$\Upsilon\colon \boldsymbol{\Gamma}^{\mathbf{Y}}_{q} f_* \xrightarrow{\sim} \mathsf{U}^{\varphi}\boldsymbol{\Gamma}^{\mathbf{X}}_{p}$$

Proof. We will start considering an object in the source that lies in the essential image of the functor $\mathsf{T} = p_*p^*$. Let $\mathscr{F} \in \mathsf{Qco}(\mathbf{X})$ with $p^*\mathscr{F} = \widetilde{M}$, for $M \in B_0$-Mod. By Proposition 5.11(*ii*), $\boldsymbol{\Gamma}^{\mathbf{X}}_{p}\mathsf{T}\mathscr{F} = (B_1 \otimes_{B_0} M, \nabla \otimes \mathrm{id})$. By 7.3

$$\mathsf{U}^{\varphi}\boldsymbol{\Gamma}^{\mathbf{X}}_{p}\mathsf{T}\mathscr{F} = (A_1 \otimes_{A_0} M, \nabla \otimes \mathrm{id})$$

On the other hand, by 2-functoriality of direct images there is a canonical isomorphism via γ_*^{-1}

$$f_*\mathsf{T}\mathscr{F} = f_*p_*p^*\mathscr{F} \xrightarrow{\sim} q_*f_{0*}\widetilde{M} = q_*\widetilde{M_{[\varphi_0]}}.$$

Therefore there is an isomorphism $\boldsymbol{\Gamma}^{\mathbf{Y}}_{q} f_*\mathsf{T}\mathscr{F} \xrightarrow{\sim} \boldsymbol{\Gamma}^{\mathbf{Y}}_{q} q_*\widetilde{M_{[\varphi_0]}}$ and we have

$$\boldsymbol{\Gamma}^{\mathbf{Y}}_{q} q_*\widetilde{M_{[\varphi_0]}} = (A_1 \otimes_{A_0} M, \nabla \otimes \mathrm{id}),$$

by Proposition 5.11(*ii*), again. This defines $\Upsilon_{\mathsf{T}\mathscr{F}}\colon \boldsymbol{\Gamma}^{\mathbf{Y}}_{q} f_*\mathsf{T}\mathscr{F} \xrightarrow{\sim} \mathsf{U}^{\varphi}\boldsymbol{\Gamma}^{\mathbf{X}}_{p}\mathsf{T}\mathscr{F}$. Notice that $\Upsilon_{\mathsf{T}(-)}$ is a natural transformation.

Finally we extend this to all quasi-coherent sheaves on $\mathbf{X}$ and all morphisms. Consider the canonical equalizer of quasi-coherent $\mathscr{O}_{\mathbf{X}}$-modules

$$\mathscr{F} \xrightarrow{\eta_{\mathscr{F}}} \mathsf{T}\mathscr{F} \underset{\mathsf{T}\eta_{\mathscr{F}}}{\overset{\eta_{\mathsf{T}\mathscr{F}}}{\rightrightarrows}} \mathsf{T}^2\mathscr{F}.$$

We may thus present every object in $\mathsf{Qco}(\mathbf{X})$ as a kernel of objects that lie in the image of the functor T. Consider the following commutative equalizer

diagram

$$\begin{tikzcd}
\boldsymbol{\Gamma}^{\mathsf{Y}}_{q} f_* \mathscr{F} \arrow[r, "\boldsymbol{\Gamma}^{\mathsf{Y}}_{q} f_* \eta_{\mathscr{F}}"] \arrow[d, dashed, "\Upsilon_{\mathscr{F}}"'] & \boldsymbol{\Gamma}^{\mathsf{Y}}_{q} f_* \mathsf{T}\mathscr{F} \arrow[r, shift left, "\boldsymbol{\Gamma}^{\mathsf{Y}}_{q} f_* \eta_{\mathsf{T}\mathscr{F}}"] \arrow[r, shift right, "\boldsymbol{\Gamma}^{\mathsf{Y}}_{q} f_* \mathsf{T}\eta_{\mathscr{F}}"'] \arrow[d, "\Upsilon_{\mathsf{T}\mathscr{F}}"'] & \boldsymbol{\Gamma}^{\mathsf{Y}}_{q} f_* \mathsf{T}^2\mathscr{F} \arrow[d, "\Upsilon_{\mathsf{T}^2\mathscr{F}}"] \\
\mathsf{U}^{\varphi}\boldsymbol{\Gamma}^{\mathsf{X}}_{p} \mathscr{F} \arrow[r, "\mathsf{U}^{\varphi}\boldsymbol{\Gamma}^{\mathsf{X}}_{p} \eta_{\mathscr{F}}"] & \mathsf{U}^{\varphi}\boldsymbol{\Gamma}^{\mathsf{X}}_{p} \mathsf{T}\mathscr{F} \arrow[r, shift left, "\mathsf{U}^{\varphi}\boldsymbol{\Gamma}^{\mathsf{X}}_{p} \eta_{\mathsf{T}\mathscr{F}}"] \arrow[r, shift right, "\mathsf{U}^{\varphi}\boldsymbol{\Gamma}^{\mathsf{X}}_{p} \mathsf{T}\eta_{\mathscr{F}}"'] & \mathsf{U}^{\varphi}\boldsymbol{\Gamma}^{\mathsf{X}}_{p} \mathsf{T}^2\mathscr{F}.
\end{tikzcd}$$

Notice that $\mathsf{U}^{\varphi}\boldsymbol{\Gamma}^{\mathsf{X}}_{p}\mathsf{T}\eta_{\mathscr{F}}\,\Upsilon_{\mathsf{T}\mathscr{F}} = \Upsilon_{\mathsf{T}^2\mathscr{F}}\,\boldsymbol{\Gamma}^{\mathsf{Y}}_{q} f_* \mathsf{T}\eta_{\mathscr{F}}$ being $\Upsilon_{\mathsf{T}(-)}$ a natural transformation. Let us check that

$$\mathsf{U}^{\varphi}\boldsymbol{\Gamma}^{\mathsf{X}}_{p}\eta_{\mathsf{T}\mathscr{F}}\,\Upsilon_{\mathsf{T}\mathscr{F}} = \Upsilon_{\mathsf{T}^2\mathscr{F}}\,\boldsymbol{\Gamma}^{\mathsf{Y}}_{q} f_* \eta_{\mathsf{T}\mathscr{F}}.$$

Applying $\boldsymbol{\Gamma}^{\mathsf{X}}_{p}$ to the map $\eta_{\mathsf{T}\mathscr{F}}\colon \mathsf{T}\mathscr{F} \to \mathsf{T}^2\mathscr{F}$, we get

$$\mathscr{F}(p_{13}, \mathsf{id})\colon \mathscr{F}(X_1, pp_2) \longrightarrow \mathscr{F}(X_1 \times_{\mathsf{X}} X, pp_3).$$

In other words, having in mind that $p_{13}w = m$ where $w\colon X_2 \to X_1 \times_{\mathsf{X}} X$ is the isomorphism in 5.7.1 and $m = \operatorname{Spec}(\nabla)$, taking $M := \mathscr{F}(X, p)$, this morphism corresponds to:

$$\nabla \otimes \mathrm{id}\colon B_1\, {}_{\eta_R}\!\otimes_{A_0} M \longrightarrow B_{1\eta_R} \otimes_{\eta_L} B_{1\eta_R} \otimes_{A_0} M.$$

Now, we apply U^{φ} and we obtain the following composition

$$A_1\, {}_{\eta_R}\!\otimes_{A_0} M \xrightarrow{\nabla\otimes\mathrm{id}} A_1\, {}_{\eta_R}\!\otimes_{\eta_L} A_1\, {}_{\eta_R}\!\otimes_{A_0} M \xrightarrow{\mathrm{id}\otimes\varphi_1\otimes\mathrm{id}} A_1\, {}_{\eta_R}\!\otimes_{\eta_L} B_{1\eta_R} \otimes_{B_0} M.$$

On the other hand, consider the 2-cartesian squares

$$\begin{tikzcd}
Y \times_{\mathsf{Y}} X \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & X \arrow[d, "fp"] \\
Y \arrow[r, "q"'] \arrow[ur, Rightarrow, "\phi_1"] & \mathsf{Y}
\end{tikzcd}
\qquad
\begin{tikzcd}
(Y \times_{\mathsf{Y}} X) \times_{\mathsf{X}} X \arrow[r, "\pi_2'"] \arrow[d, "\pi_1'"'] & X \arrow[d, "p"] \\
Y \times_{\mathsf{Y}} X \arrow[r, "p\pi_2"'] \arrow[ur, Rightarrow, "\phi_2"] & \mathsf{X}
\end{tikzcd}$$

and the morphism $l\colon (Y \times_{\mathsf{Y}} X) \times_{\mathsf{X}} X \longrightarrow Y \times_{\mathsf{Y}} X$. satisfying that $\pi_1 l = \pi_1 \pi_1'$, $\pi_2 l = \pi_2'$ and $\phi_1 l = f\phi_2 \circ \phi_1 \pi_1'$. Applying $\boldsymbol{\Gamma}^{\mathsf{Y}}_{q} f_*$ to $\eta_{\mathsf{T}\mathscr{F}}$, we have the following commutative diagram

$$\begin{tikzcd}
\boldsymbol{\Gamma}^{\mathsf{Y}}_{q} f_* \mathsf{T}\mathscr{F} \arrow[r, "\boldsymbol{\Gamma}^{\mathsf{Y}}_{q} f_* \eta_{\mathsf{T}\mathscr{F}}"] \arrow[d, "\text{via } \gamma_*"'] & \boldsymbol{\Gamma}^{\mathsf{Y}}_{q} f_* \mathsf{T}^2\mathscr{F} \arrow[d, "\text{via } \gamma_*"] \\
\mathscr{F}(Y \times_{\mathsf{Y}} X, p\pi_2) \arrow[r, "\mathscr{F}(l, \mathsf{id})"] & \mathscr{F}((Y \times_{\mathsf{Y}} X) \times_{\mathsf{X}} X, p\pi_2')
\end{tikzcd}$$

Consider now the diagram

$$\begin{tikzcd}
(Y \times_{\mathsf{Y}} X) \times_{\mathsf{X}} X \arrow[r, "l"] & Y \times_{\mathsf{Y}} X \\
Y_{1\,p_2}\!\times_{f_0p_1} X_1 \arrow[r, "l'"] \arrow[u, "c'"] & Y_{1\,p_2}\!\times_Y X \arrow[u, "c"']
\end{tikzcd}$$

where c is the composition of the following isomorphisms

$$Y_{1\,p_2}\!\times_Y X = (Y \times_{\mathsf{Y}} Y)_{p_2}\!\times_Y X \xrightarrow{\sim} Y_{\;q}\!\times_{qf_0} X \xrightarrow{\sim} Y \times_{\mathsf{Y}} X,$$

c' is the isomorphism given by

$$Y_{1\,p_2}\!\times_{f_0p_1} X_1 \xrightarrow{\sim} (Y_{1\,p_2}\!\times_{f_0} X) \times_{\mathsf{X}} X \xrightarrow{c\times\mathrm{id}} (Y \times_{\mathsf{Y}} X) \times_{\mathsf{X}} X$$

and $l' = c^{-1} l c'$. A computation shows that the following composition

$$\begin{aligned}
Y_{1\,p_2}\!\times_{f_0p_1} X_1 &\xrightarrow{\sim} Y_{1\,p_2}\!\times_{f_0p_1} X_{1\,p_2}\!\times_X X \\
&\xrightarrow{\mathrm{id}\times f_1\times\mathrm{id}} Y_{1\,p_2}\!\times_{p_1} Y_{1\,p_2}\!\times_Y X \\
&\xrightarrow{s\times_Y X} (Y_{1\,p_1}\!\times_{p_2} Y_1)\times_Y X \\
&\xrightarrow{m\times\mathrm{id}} Y_{1\,p_2}\!\times_Y X
\end{aligned}$$

agrees with l', where s is the isomorphism that interchanges the factors of the fiber product. But then, considering the isomorphisms $Y_1 \times_Y X = \operatorname{Spec}(A_{1\;\eta_R}\!\otimes_{A_0} B_0)$ and $Y_1 \times_Y X_1 = \operatorname{Spec}(A_{1\;\eta_R}\!\otimes_{\eta_L} B_1)$, l' is identified the spectrum of the composition

$$A_{1\;\eta_R}\!\otimes_{A_0} B_0 \xrightarrow{\nabla\otimes\mathrm{id}} A_{1\;\eta_R}\!\otimes_{\eta_L} A_{1\;\eta_R}\!\otimes_{A_0} B_0 \xrightarrow{\mathrm{id}\otimes\varphi_1\otimes\mathrm{id}} A_{1\;\eta_R}\!\otimes_{\eta_L} B_{1\eta_R}\otimes_{B_0} B_0.$$

Finally, as the map $\mathscr{F}(l', \mathsf{id})$ corresponds to the last map tensored with $M = \mathscr{F}(X, p)$ this shows the agreement of both morphisms.

The fact that the rows are equalizers defines the promised dashed natural isomorphism $\Upsilon_{\mathscr{F}}$. □

Corollary 7.7. *There is a natural isomorphism of functors*

$$\boldsymbol{\Gamma}^{\mathsf{X}}_p f^* \xrightarrow{\sim} B_0 \otimes_{A_0} \boldsymbol{\Gamma}^{\mathsf{Y}}_q.$$

Proof. Denote by Θ^{Y}_q a quasi-inverse of $\boldsymbol{\Gamma}^{\mathsf{Y}}_q$. The functor $\boldsymbol{\Gamma}^{\mathsf{X}}_p f^* \Theta^{\mathsf{Y}}_q$ is left adjoint to U^{φ}, so it has to agree with $B_0 \otimes_{A_0} -$. □

8. Deligne-Mumford stacks and functoriality for the étale topology

In section 6 we have shown how to obtain an adjoint functoriality of the category of quasi-coherent sheaves on a geometric stack overcoming the difficulty of the lack of functoriality of the fppf topos. But one can (and should) wonder how this construction is related to the case where a functorial topos exists. This is the case for the étale topos on a Deligne-Mumford stack, as follows from Theorem 8.7 below. We recall that the corresponding theory of

quasi-coherent sheaves is equivalent to the one in the Zariski topology whenever the stacks are equivalent to schemes [SGA 4_2, VII, 4.3]. The answer is that our construction agrees with it. Let us see how.

8.1. Let X be a geometric Deligne-Mumford stack and consider the category $\mathsf{Aff}_{\text{ét}}/\mathsf{X}$ whose objects are pairs (U,u) with U an affine scheme and $u: U \to \mathsf{X}$ an étale morphism, morphisms are as in Aff/X. Note that all morphisms are étale. We define a topology in this category having as coverings finite families of étale morphisms that are jointly surjective. We denote also by $\mathsf{Aff}_{\text{ét}}/\mathsf{X}$ this site and by $\mathsf{X}_{\text{ét}}$ the associated topos of sheaves. The category $\mathsf{Aff}_{\text{ét}}/\mathsf{X}$ is ringed by the sheaf of rings $\mathscr{O}\colon \mathsf{Aff}_{\text{ét}}/\mathsf{X} \to \mathsf{Ring}$, defined by $\mathscr{O}(U,u) = B$ with $U = \operatorname{Spec}(B)$. We define

$$\mathsf{Qco}_{\text{ét}}(\mathsf{X}) := \mathsf{Qco}(\mathsf{Aff}_{\text{ét}}/\mathsf{X}, \mathscr{O}).$$

As in Theorem 3.11, $\mathsf{Qco}_{\text{ét}}(\mathsf{X})$ agrees with the category of Cartesian sheaves, and we will use this fact freely.

Let $f: \mathsf{X} \to \mathsf{Y}$ be a 1-morphism of geometric *Deligne-Mumford* stacks.

8.2. Direct image for the étale topology. Let $(V,v) \in \mathsf{Aff}_{\text{ét}}/\mathsf{Y}$. Denote by $\mathbf{J}_{\text{ét}}(v,f)$ the category whose objects are the 2-commutative squares

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[d, "u"'] & V \arrow[d, "v"] \\ \mathsf{X} \arrow[r, "f"'] \arrow[ur, Rightarrow, "\gamma"] & \mathsf{Y} \end{tikzcd} \tag{8.2.1}$$

where $(U,u) \in \mathsf{Aff}_{\text{ét}}/\mathsf{X}$. As in 6.1 (with appropriate changes) we define for $\mathscr{G} \in \mathsf{Pre}(\mathsf{Aff}_{\text{ét}}/\mathsf{X})$,

$$(f_*^{\text{ét}}\mathscr{G})(V,v) = \varprojlim_{\mathbf{J}_{\text{ét}}(v,f)} \mathscr{G}(U,u).$$

Again, if f is an affine morphism, then $(f_*^{\text{ét}}\mathscr{G})(V,v) = \mathscr{G}(V \times_{\mathsf{Y}} \mathsf{X}, p_2)$.

8.3. Inverse image for the étale topology. Now, fix $(U,u) \in \mathsf{Aff}_{\text{ét}}/\mathsf{Y}$ and denote by $\mathbf{I}_{\text{ét}}(u,f)$ the category whose objects are the 2-commutative squares as in diagram (8.2.1), with $(V,v) \in \mathsf{Aff}_{\text{ét}}/\mathsf{Y}$. By a similar procedure to 6.2 define for $\mathscr{F} \in \mathsf{Pre}(\mathsf{Aff}_{\text{ét}}/\mathsf{Y})$ and $(U,u) \in \mathsf{Aff}_{\text{ét}}/\mathsf{X}$

$$(f^{\mathsf{p}}\mathscr{F})(U,u) = \varinjlim_{\mathbf{I}_{\text{ét}}(u,f)} \mathscr{F}(V,v).$$

Notice that if f is étale, then $(f^{\mathsf{p}}\mathscr{F})(U,u) = \mathscr{F}(U, fu)$.

Proposition 8.4. *The previous constructions define a pair of adjoint functors*

$$\mathsf{Pre}(\mathsf{Aff}_{\text{ét}}/\mathsf{X}) \underset{f_*^{\text{ét}}}{\overset{f_{\text{ét}}^{\mathsf{p}}}{\rightleftarrows}} \mathsf{Pre}(\mathsf{Aff}_{\text{ét}}/\mathsf{Y}).$$

Proof. The proof is completely similar to the one in Proposition 6.5. ☐

Proposition 8.5. *The functor* $f^{\mathsf{p}}\colon \mathsf{Pre}(\mathsf{Aff}_{\text{ét}}/\mathbf{Y}) \to \mathsf{Pre}(\mathsf{Aff}_{\text{ét}}/\mathbf{X})$ *is* exact.

Proof. The functor f^{p} is right exact, because it has a right adjoint. To prove that f^{p} is left exact it is enough to show that $\mathbf{I}_{\text{ét}}(u,f)$ is a cofiltered category for $(U,u) \in \mathsf{Aff}_{\text{ét}}/\mathbf{Y}$.

By [SGA 4_1, I, 2.7] we have to prove that the category $\mathbf{I}_{\text{ét}}(u,f)$ is connected and satisfies PS 1) and PS 2) in *loc. cit.* The category $\mathbf{I}_{\text{ét}}(u,f)$ is connected since it has finite products (analogous to the fppf case, Lemma 6.12).

To prove PS 1), for $i \in \{1,2\}$ take $(j_i,\beta_i)\colon (V_i,v_i,h_i,\gamma_i) \to (V,v,h,\gamma)$ morphisms in $\mathbf{I}_{\text{ét}}(u,f)$, the projections $p_i\colon V_1 \times_V V_2 \to V_i$ and $h'\colon U \to V_1 \times_V V_2$ the morphism verifying $p_i h' = h_i$. We have that $(V_1 \times_V V_2, v_1 p_1, h', \gamma_1)$ is in $\mathbf{I}_{\text{ét}}(u,f)$ and the morphisms (p_1,id) and $(p_2, \beta_2^{-1} p_2 \circ \beta_1 p_1)$ satisfy

$$(j_1,\beta_1)\circ(p_1,\mathrm{id}) = (j_2,\beta_2)\circ(p_2,\beta_2^{-1}p_2\circ\beta_1 p_1).$$

Finally, we prove PS 2): Let $(g,\alpha),(g',\alpha')\colon (V,v,h,\gamma)\to(V',v',h',\gamma')$ be morphisms in $\mathbf{I}_{\text{ét}}(u,f)$ and consider the pull-back

$$\begin{tikzcd} V' \times_{\mathbf{Y}} V' \arrow[r, "p_2"] \arrow[d, "p_1"'] & V' \arrow[d, "v'"] \\ V' \arrow[r, "v'"] \arrow[ur, Rightarrow, "\beta"] & \mathbf{Y} \end{tikzcd}$$

We have the morphism $r\colon V \to V' \times_{\mathbf{Y}} V'$ verifying $p_1 r = g$, $p_2 r = g'$ and $\beta r = \alpha' \circ \alpha^{-1}$. Consider the Cartesian square

$$\begin{tikzcd} Z \arrow[r, "q_2"] \arrow[d, "q_1"'] & V' \arrow[d, "\delta"] \\ V \arrow[r, "r"] & V' \times_{\mathbf{Y}} V' \end{tikzcd}$$

and the morphism $l\colon U \to Z$ such that $q_1 l = h$ and $q_2 l = h'$. Notice that q_1 is étale because δ is an open embedding and therefore, étale. The object $(Z, v q_1, l, \gamma) \in \mathbf{I}_{\text{ét}}(u,f)$ and the morphism $(q_1,\mathrm{id})\colon (Z,vq_1,l,\gamma) \to (V,v,h,\gamma)$ is the equalizer of (g,α) and (g',α'). □

Remark. Notice that this result, specifically PS 2), is *false* for finer topologies.

Proposition 8.6. *If* $\mathscr{F}$ *is a sheaf on* $\mathsf{Aff}_{\text{ét}}/\mathbf{X}$*, then* $f_*^{\text{ét}}\mathscr{F}$ *is a sheaf.*

Proof. It can be proven similarly to Proposition 6.7. □

Theorem 8.7. *Let* $f\colon \mathbf{X} \to \mathbf{Y}$ *be a 1-morphism of geometric Deligne-Mumford stacks, there is an associated morphism of topos*

$$f_{\text{ét}}\colon \mathbf{X}_{\text{ét}} \longrightarrow \mathbf{Y}_{\text{ét}}$$

Proof. Let $\mathscr{G} \in \mathbf{Y}_{\text{ét}}$, we denote by $f_{\text{ét}}^{-1}\mathscr{G}$ the sheaf associated to the presheaf $f^{\mathsf{p}}\mathscr{G}$. This, together with Proposition 8.6 provides an adjunction

$$\mathbf{X}_{\text{ét}} \underset{f_*^{\text{ét}}}{\overset{f_{\text{ét}}^{-1}}{\leftrightarrows}} \mathbf{Y}_{\text{ét}}$$

The pair $(f_*^{\text{ét}}, f_{\text{ét}}^{-1})$ defines a morphism of topoi, since $f_{\text{ét}}^{-1}$ is exact because so is f^{p} and the sheafification functor. □

Proposition 8.8. *Let $f\colon \mathbf{X} \to \mathbf{Y}$ and $g\colon \mathbf{Y} \to \mathbf{Z}$ be 1-morphisms of geometric Deligne-Mumford stacks. It holds that $(gf)_*^{\text{ét}} \cong g_*^{\text{ét}} f_*^{\text{ét}}$*

Proof. The proof is *mutatis mutandis* the same as Proposition 6.9. □

Corollary 8.9. *In the previous situation $(gf)_{\text{ét}}^{-1} \xrightarrow{\sim} f_{\text{ét}}^{-1} g_{\text{ét}}^{-1}$.*

Proof. It is is a consequence of the previous proposition by adjunction. □

8.10. Let $f\colon \mathbf{X} \to \mathbf{Y}$ be a 1-morphism of geometric Deligne-Mumford stacks. The ring homomorphism $f_{\text{ét}}^{\#}\colon \mathscr{O}_{\mathbf{Y}} \to f_*\mathscr{O}_{\mathbf{X}}$ induces a canonical morphism of ringed topoi

$$(f_{\text{ét}}, f_{\#}^{\text{ét}})\colon (\mathbf{X}_{\text{ét}}, \mathscr{O}_{\mathbf{X}}) \longrightarrow (\mathbf{Y}_{\text{ét}}, \mathscr{O}_{\mathbf{Y}})$$

where $f_{\#}^{\text{ét}}\colon f^{-1}\mathscr{O}_{\mathbf{Y}} \to \mathscr{O}_{\mathbf{X}}$ is the morphism adjoint to $f_{\text{ét}}^{\#}$. Now we can define the functor $f_{\text{ét}}^{*}\colon \mathscr{O}_{\mathbf{Y}}\text{-}\mathsf{Mod} \to \mathscr{O}_{\mathbf{X}}\text{-}\mathsf{Mod}$ by the usual formula

$$f_{\text{ét}}^{*}\mathscr{F} := \mathscr{O}_{\mathbf{X}} \otimes_{f_{\text{ét}}^{-1}\mathscr{O}_{\mathbf{Y}}} f_{\text{ét}}^{-1}\mathscr{F}.$$

It is left adjoint to $f_*^{\text{ét}}\colon \mathscr{O}_{\mathbf{X}}\text{-}\mathsf{Mod} \to \mathscr{O}_{\mathbf{Y}}\text{-}\mathsf{Mod}$.

Proposition 8.11. *In the previous situation, we have*

(*i*) *if $\mathscr{G} \in \mathsf{Qco}_{\text{ét}}(\mathbf{Y})$ then $f_{\text{ét}}^{*}\mathscr{G} \in \mathsf{Qco}_{\text{ét}}(\mathbf{X})$, and*
(*ii*) *if $\mathscr{F} \in \mathsf{Qco}_{\text{ét}}(\mathbf{X})$ then $f_*^{\text{ét}}\mathscr{F} \in \mathsf{Qco}_{\text{ét}}(\mathbf{Y})$.*

Proof. This can be shown in a similar way to Propositions 6.16 and 6.17. □

Corollary 8.12. *There is a pair of adjoint functors*

$$\mathsf{Qco}_{\text{ét}}(\mathbf{X}) \underset{f_*^{\text{ét}}}{\overset{f_{\text{ét}}^{*}}{\leftrightarrows}} \mathsf{Qco}_{\text{ét}}(\mathbf{Y}).$$

Proof. Combine the previous proposition with 8.10. □

For $\mathbf{X}$ a geometric Deligne-Mumford stack we denote by $\mathsf{r}_{\mathbf{X}} : \mathbf{X}_{\text{fppf}} \to \mathbf{X}_{\text{ét}}$ the restriction functor.

Proposition 8.13. *Let $f\colon \mathbf{X} \to \mathbf{Y}$ be a 1-morphism of geometric Deligne-Mumford stacks. There is an isomorphism of functors from $\mathbf{X}_{\text{fppf}}$ to $\mathbf{Y}_{\text{ét}}$ $\mathsf{r}_{\mathbf{Y}} f_* \cong f_*^{\text{ét}} \mathsf{r}_{\mathbf{X}}$.*

Proof. As p is an affine morphism, then $\mathsf{r}_{\mathbf{X}} p_* = p_*^{\text{ét}} \mathsf{r}_X$. In general, let $\mathscr{F} \in \mathbf{X}_{\text{fppf}}$. Recall that we have an equalizer in $\mathbf{X}_{\text{fppf}}$

$$\mathscr{F} \xrightarrow{\eta_{\mathscr{F}}} \mathsf{T}\mathscr{F} \underset{\eta_{\mathsf{T}\mathscr{F}}}{\overset{\mathsf{T}\eta_{\mathscr{F}}}{\rightrightarrows}} \mathsf{T}^2\mathscr{F}, \tag{8.13.1}$$

with $\mathsf{T} = p_*p^*$. By Proposition 6.9 and Proposition 8.8, using also that fp is an affine morphism we have that

$$\mathsf{r}_{\mathbf{Y}} f_* p_* \cong \mathsf{r}_{\mathbf{Y}} (fp)_* = (fp)^{\text{ét}}_* \mathsf{r}_X \cong f^{\text{ét}}_* p^{\text{ét}}_* \mathsf{r}_X$$

From this it follows that $\mathsf{r}_{\mathbf{Y}} f_* \mathsf{T}\mathscr{F} = f^{\text{ét}}_* \mathsf{r}_{\mathbf{X}} \mathsf{T}\mathscr{F}$ and we construct the diagram whose rows are equalizers

$$\begin{tikzcd}
\mathsf{r}_{\mathbf{Y}} f_*\mathscr{F} \arrow[r] \arrow[d, dashed, "\wr"] & \mathsf{r}_{\mathbf{Y}} f_*\mathsf{T}\mathscr{F} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\wr"] & \mathsf{r}_{\mathbf{Y}} f_*\mathsf{T}^2\mathscr{F} \arrow[d, "\wr"] \\
f^{\text{ét}}_* \mathsf{r}_{\mathbf{X}}\mathscr{F} \arrow[r] & f^{\text{ét}}_* \mathsf{r}_{\mathbf{X}}\mathsf{T}\mathscr{F} \arrow[r, shift left] \arrow[r, shift right] & f^{\text{ét}}_* \mathsf{r}_{\mathbf{X}}\mathsf{T}^2\mathscr{F}
\end{tikzcd}$$

that shows existence of the dashed isomorphism. Notice that this isomorphism is clearly compatible with morphisms. □

Notice that if $\mathscr{G} \in \mathsf{Qco}(\mathbf{X})$, then $\mathsf{r}_{\mathbf{X}}\mathscr{G} \in \mathsf{Qco}_{\text{ét}}(\mathbf{X})$, an obvious fact if we consider the Cartesian characterization. We also denote by $\mathsf{r}_{\mathbf{X}}$ the induced functor between the categories of quasi-coherent sheaves.

Proposition 8.14. *The functor* $\mathsf{r}_{\mathbf{X}} \colon \mathsf{Qco}(\mathbf{X}) \to \mathsf{Qco}_{\text{ét}}(\mathbf{X})$ *is an equivalence of categories.*

Proof. Let $\mathsf{Desc}^{\text{ét}}_{\mathsf{qc}}(X/\mathbf{X})$ be the category whose objects are the pairs $(\mathscr{G}, t)$, where $\mathscr{G} \in \mathsf{Qco}_{\text{ét}}(X)$ and $t \colon p^*_{2,\text{ét}}\mathscr{G} \to p^*_{1,\text{ét}}\mathscr{G}$ is an isomorphism verifying the usual cocycle condition. An argument similar to the one that proves Proposition 4.6 yields an equivalence of categories

$$\mathsf{D}_{\text{ét}} \colon \mathsf{Qco}_{\text{ét}}(\mathbf{X}) \longrightarrow \mathsf{Desc}^{\text{ét}}_{\mathsf{qc}}(X/\mathbf{X}).$$

Indeed, set $X_1 := X \times_{\mathbf{X}} X$. We have natural isomomorphisms of functors

$$w_i \colon p_i^* \left(\widetilde{\mathscr{G}(X)}\right) \longrightarrow \widetilde{p_i^*\mathscr{G}(X_1)}, \qquad i \in \{1,2\}$$

defined, for $(U,u) \in \mathsf{Aff}_{\mathsf{fppf}}/X$, with $U = \operatorname{Spec}(B)$ and $u = \operatorname{Spec}(v)$ by

$$w_i(U,u) \colon B \otimes_{A_0} \mathscr{G}(X) \xrightarrow{\sim} B\,{}_{v\epsilon}\!\otimes_{A_1} A_{1\eta_L} \otimes_{A_0} \mathscr{G}(X) \xrightarrow{\mathrm{id} \otimes \mathscr{G}(p_i)^{\mathsf{a}}} B\,{}_{v\epsilon}\!\otimes_{A_1} \mathscr{G}(X_1, p_i)$$

with $i \in \{1,2\}$. The induced functor

$$\mathsf{r}_{X/\mathbf{X}} \colon \mathsf{Desc}_{\mathsf{qc}}(X/\mathbf{X}) \longrightarrow \mathsf{Desc}^{\text{ét}}_{\mathsf{qc}}(X/\mathbf{X})$$

given by $\mathsf{r}_{X/\mathbf{X}}(\mathscr{G}, t) = (\mathsf{r}_X\mathscr{G}, \mathsf{r}_{X_1} t)$ is also an equivalence of categories; a quasi-inverse of $\mathsf{r}_{X/\mathbf{X}}$ is the functor $\mathsf{W}_{X/\mathbf{X}} \colon \mathsf{Desc}^{\text{ét}}_{\mathsf{qc}}(X/\mathbf{X}) \to \mathsf{Desc}_{\mathsf{qc}}(X/\mathbf{X})$ given by

$$\mathsf{W}_{X/\mathbf{X}}(\mathscr{G}, t) = (\widetilde{\mathscr{G}(X)}, w_1^{-1}(t(\widetilde{X_1, \mathrm{id}_{X_1}}))w_2)$$

The result follows from the fact that the following diagram

$$\begin{tikzcd} \mathsf{Qco}(\mathbf{X}) \arrow[r, "\mathsf{r}_{\mathbf{X}}"] \arrow[d, "\mathsf{D}"'] & \mathsf{Qco}_{\text{ét}}(\mathbf{X}) \arrow[d, "\mathsf{D}_{\text{ét}}"] \\ \mathsf{Desc}_{\mathsf{qc}}(X/\mathbf{X}) \arrow[r, "\mathsf{r}_{X/\mathbf{X}}"] & \mathsf{Desc}^{\text{ét}}_{\mathsf{qc}}(X/\mathbf{X}) \end{tikzcd}$$

commutes because $\mathsf{r}_X p^*\mathscr{G} = p^*_{\text{ét}}\mathsf{r}_{\mathbf{X}}\mathscr{G}$. □

Remark. The same strategy used in Propositions 8.13 and 8.14 may be used to prove that on an ordinary scheme the quasi-coherent sheaves over the Zariski topology agree with those defined like here on the étale topology. We leave the details to the interested readers.

Corollary 8.15. *In the previous setting, there is a natural isomorphism of functors from* $\mathsf{Qco}(\mathbf{Y})$ *to* $\mathsf{Qco}_{\text{ét}}(\mathbf{X})$

$$\mathsf{r}_{\mathbf{X}} f^* \xrightarrow{\sim} f^*_{\text{ét}}\mathsf{r}_{\mathbf{Y}}.$$

Proof. If the functor $\mathsf{s}_{\mathbf{Y}} \colon \mathsf{Qco}_{\text{ét}}(\mathbf{Y}) \to \mathsf{Qco}(\mathbf{Y})$ is a quasi-inverse of $\mathsf{r}_{\mathbf{Y}}$, then the functor $\mathsf{r}_{\mathbf{X}} f^* \mathsf{s}_{\mathbf{Y}}$ is left adjoint to $f^{\text{ét}}_*$. □

Remark. The previous discussion expresses the fact that the formalism of functoriality that we established in §6 agrees with the usual formalism derived from the étale topos map induced from a 1-morphism of Deligne-Mumford stacks on the category of quasi coherent sheaves.

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